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Degree: Master of Science

Year this Degree Granted: 2001

University of Alberta

Mathematical Model Of An Agricultural-Industrial-Ecospheric System

by

Ibrahim Agyemang

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics

Department of Mathematical Sciences

Edmonton, Alberta

Fall 2001

UNIVERSITY OF ALBERTA

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Mathematical Model of an Agricultural-Industrial-Ecospheric System** submitted by **Ibrahim Agyemang** in partial fulfillment of the requirements for the degree of **Master of Science** in Applied Mathematics.

Dedication

This thesis is dedicated to the memory of:

Mariyam Pinamang Agyemang (Mother, deceased)

and

Ghulam Ahmad Agyemang (Brother, deceased)

whose inspirational conversation, tender love, advice, care and companionships are greatly missed. May Allah out of His grace have mercy on their souls, Amen.

ABSTRACT

We model long term trajectories for the shares of agricultural, industrial and ecospheric assets in an open interacting economic system, using a three dimensional system of nonlinear ordinary differential equations based on a predator-prey paradigm. For the two dimensional case, corresponding to an ecosphere in equilibrium, the system is analyzed using the techniques of linearized stability and Liapunov stability theory. Explicit mathematical criteria are formulated for the extinction of industrial assets and the persistence of both industrial and agricultural assets.

For the three dimensional case, using a constant rate of correction for the ecosphere, the system is analyzed using techniques of linearized stability theory and the Routh-Hurwitz criterion.

Finally, we introduce a hysteresis term into the ecospheric equation of the three dimensional system.

In each of the above cases, numerical examples and the associated graphs are exhibited for various parametric configurations. Results for the hysteresis case are compared with the non-hysteresis case.

Acknowledgements

All Praise are due Allah, the most high, the guardian and the guide, the bestower of favor, honor and security, the protector and the opener of doors of success for mankind.

I wish to express my sincere thanks and gratitude to my supervior, Prof. Jack W. Macki and co-supervisor, Prof. H.I. Freedman for their invaluable guidence and time spent in assisting me in preparing this thesis; Without them life in the department of mathematical sciences would have been an illusion.

I also extend my appreciation to Dr. Gerda De Vries for all the assistance offered me as my teaching mentor and advice on other academic related issues. Thanks go to Dr. D.P Wiens and Dr. J. Tuszynski for their academic advice and support.

I wish to thank Dr. G.E. Swaters, Dr. Riemenschneider, Dr. J.W. Macki, Dr. H.I. Freedman, and Dr. M. Li for providing me with financial assistance for my research. Thanks to the Department of mathematical sciences for offering me admission and financial assistance for my masters program.

Special thanks go to Marion Benedict, Laura Heiland, Dona Guelzow and all other support staff and academic staff of the mathematical sciences department.

I also wish to express my profound appreciation to Mr. Emmanuel Wiredu, collegues and all Ghanaian students for making me feel welcome at home in Edmonton, Alberta.

To the following family members of mine, Mr.Eassah Agyemang (father), Medina Pinamang (step mother), Mohammed and Nusrat(Emelia), Adam and Brenda Makaya, Aysha and Dawood, Ahmad and Hannah, Mohammed(Osofo) and Helima, Malik, Fati and Suleman Kofi, Suleman, Mariam, Yusif, Hawa, Helima, Sakeena and Omar, I say May Allah bless you for all you have done for me.

Contents

Chapter 1

Introduction

1.1 Overview of the Problem

Agriculture faces the problem of sustainability worldwide. The evidence for this lies in part in the following facts:
(i) the extensive and expensive subsidies to agriculture in the developed countries, and the extreme poverty and ecospheric degradation in the developing countries [23],
(ii) a comparative study of farms in the United States and Canada in 1991 revealed that virtually all North American farms incorporate off-farm income in their business strategies to manage uncertainties and, in most cases, bring net margins up beyond the poverty line ([23],[24]).

In this thesis, we are basically interested in studying and investigating the effects of ecospheric degradation on agriculture and industry, and the future of agriculture and industry as we continue to degrade the ecosphere and government support in the form of subsidies to agriculture is reduced due to government deficits and economic recession.

We will be using the predator-prey paradigm to model the interaction between agriculture, industry and the ecosphere. The predator-prey paradigm is used because it has extensively been used in modelling ecological interactions [10], in some cases with success. For example, in 1994, Apedaile et al [2] modelled the interaction between agriculture, industry and the ecosphere using the predator-prey model. They assumed that the ecosphere was in a state of equilibrium and determined the stability properties and bifurcation behavior of the resulting two dimensional system. They predicted that agricultural economy ages by slowing its learning, seeking and obtaining greater stability, and eventually leads to trapping. In 1997 Solomonovich et al [23] adapted a minimum safe standard policy for the ecosphere. Later in 1998, Solomonovich et al [24], motivated by an earlier work of Samuelson [22] which used a Lotka-Volterra predation model, tried to address the question of what happens when the ecosphere is not sustained. They modelled the agriculture-industry-ecosphere interactions to include natural recovery and degradation. They showed that it was possible to eliminate uncertainty and replace it with predictablity by simply modifying some of the model parameters.

Our main aim in this thesis is to model the agriculture-industry-ecosphere interactions using a simpler predator-prey model which uses a smaller number of parameters and also to see how far we can go beyond the works of Solomonovich et al and Apedaile et al by introducing a hysteresis term in the ecospheric equation of the model.

1.2 Basic Definitions

In this section we define the key words and phrases (some of which we have encountered already) used in the forthcoming chapters.

Agriculture [18]:

Agriculture is the occupation, business or science of cultivating the land, producing crops and raising animals. We will however generalize agriculture to include all renewable resource sectors and 'harvesting' agents of an industrial economy in particular (see [2], [24]).

Industry [18]:

Industry is an organized economic activity connected with production, manufacturing, or construction of a particular product or range of products, especially the ones that have become excessively commercialized or standardized.

Ecosphere or Environment [18]:

This refers to the natural world, within which people, animals and plants live. This includes the whole complex of climatic, edaphic and biotic factors, such as light or food supply, that influence the life of an organism or an ecological community.

Asset [18]:

An asset is something that is useful and contributes to the success of an activity. These include cash value or the property of something to which a value can be assigned, etc.

Steady State ([19],[21],[24]):

Let

$$\frac{dX}{dt} = f(X) \tag{1.1}$$

be a (non)linear system in $\Re^n$. Let G be an open subset of $\Re^n$ containing X^*. Then

X^* is said to be a steady state of Equation (1.1) if $f(X^*) = 0$.

Jacobian Matrix J(X):

The Jacobian matrix J(X) of Equation (1.1) is defined as

$$J(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(X) & . \ . \ . & \frac{\partial f_1}{\partial x_n}(X) \\ . & & \\ . & & \\ . & & \\ \frac{\partial f_n}{\partial x_1}(X) & . \ . \ . & \frac{\partial f_n}{\partial x_n}(X) \end{bmatrix}. \tag{1.2}$$

Differentiability ([21],[14]):

The function $f : \Re^n \to \Re^n$ is said to be differentiable at $x_0 \in \Re^n$ if $J(x_0)$ satisfies $\lim_{||h|| \to 0} \frac{||f(x_0+h) - f(x_0) - J(x_0)h||}{||h||} = 0$, where $||.||$ denotes the norm on $\Re^n$.

Continuity ([21],[14]):

Let $||.||$ be the norm on $\Re^n$. Then $f : \Re^n \to \Re^n$ is said to be continuous at $x_0 \in \Re^n$ if for all $\epsilon > 0$ there exists a $\delta' > 0$ such that $||x - x_0|| < \delta'$ implies that $||f(x) - f(x_0)|| < \epsilon$.

If f is continuous at each point $x \in G$, where G is an open subset of $\Re^n$, then we say f is continuous on $G \subset \Re^n$ and we write $f \in C(G)$.

Continuous Differentiability ([21],[14]):

If $f : G \to \Re^n$ is differentiable on G and the Jacobian J(X) is continuous on G, then we say f is continuously differentiable function on G and write $f \in C'(G)$.

Flow of a Differential Equation ([21],[12]):

Let G be an open subset of $\Re^n$ and $f \in C'(G)$. For $X_0 \in G$, let $\Phi(t, X_0) = \Phi_t(X)$ be the solution of Equation (1.1) defined on its maximal interval of existence. Then Φ_t is called the flow of Equation (1.1).

Stability of Steady States ([21],[12]):

Suppose X^* is a steady state of Equation (1.1). Then X^* is said to be stable if for all $\epsilon > 0$ there exists a $\delta' > 0$ such that for all X in the δ'-neighborhood of X^* and $t \geq 0$, we have $\Phi_t(X)$ in the ϵ-neighborhood of X^*.

The steady state X^* is said to be unstable if it is not stable.

The steady state X^* is said to be asymptotically stable if it is stable and there exists a $\delta' > 0$ such that for all X in the the δ'-neighborhood of X^*, we have $\lim_{t\to\infty} \Phi_t(X) = X^*$.

1.3 Symbols and Notation

In this section we introduce some of the symbols and notation we will be using in the forthcoming chapters.

Symbol	Description
A(t)	Agricultural assets at anytime t
I(t)	Industrial assets at anytime t
E(t)	Ecospheric assets at anytime t
α	Maximum growth rate for agricultural assets
β	Diminishing returns coefficient for agriculture
γ	Terms of trade between agriculture and industry
ξ	Constant depreciation rate of industry
η	Linear depreciation rate of industry
δ	Growth rate for industry
κ	Maximum degradation rate for the ecosphere
ϑ	Natural recovery rate for the ecosphere

μ	Correction rate for the ecosphere
k	A non-negative integer
λ_k	Eigenvalue of a matrix
F_k	Steady states for the two dimensional system
J_k	Jacobian matrix corresponding to F_k
F_{kA}	Axial steady states for the three dimensions system
J_{kA}	Jacobian matrix corresponding to F_{kA}
F_{kP}	Planar steady states for the three dimensional system
J_{kP}	Jacobian matrix corresponding to F_{kP}
F_{kI}	Interior steady states for the three dimensions system
J_{kI}	Jacobian matrix corresponding to F_{kI}
$E_{\max}$	Maximum threshold value for the ecosphere
$E_{\min}$	Minimum threshold value for the ecosphere
E_{med}	Critical threshold for the ecosphere
T_k	Time at which E(t) passes the k^{th} threshold
G	An open subset of $\Re^n$
$C'(G)$	Continuously differentiable functions on G
$V(.)$	Liapunov function

The following notations are also used:

$$E^* = E^*_{1I} = \frac{\beta\vartheta + \sqrt{\beta^2\vartheta^2 + 4\mu\beta(\beta\vartheta + \kappa\alpha)}}{2(\beta\vartheta + \kappa\alpha)}$$

$$E^{\pm}_{2I} = \frac{(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi \pm \sqrt{[(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi]^2 + 4\mu(\beta\eta - \gamma\delta)[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha]}}{2[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha]}$$

1.4 Epilogue to the Introduction

We emphasize once again that the idea to formulate a mathematical model using the predator-prey paradigm for the study and analysis of the interaction between agriculture, industry and the ecosphere is not new. Apedaile et al [2] and Solomonivich et al ([23],[24]) did some work in this direction. We just want to see how far we can go beyond their work.

In the forthcoming chapters, agriculture-industry-ecosphere interactions are modelled using the predator-prey paradigm. The chapters are structured as follows: Chapter 2 deals with interactions between agriculture and industry assuming that the ecosphere is in a state of equilibrium. Stability analyses are performed and a criterion for global stability is given. In Chapter 3, we investigate the consequences of introducing the ecospheric equation into the agricultural-industrial model by assuming that there is always a constant effort to correct the loss of ecospheric assets as a result of its interaction with agriculture. We study the local stability conditions of the steady states. In Chapter 4, we consider the model from Chapter 3 but this time assume that the effort is not a constant, but instead dependent on the state of the ecosphere and some thresholds. Numerical examples will be used to illustrate the results, and this will be done throughout the thesis.

Chapter 2

Equilibria and Dynamics in an Agricultural-Industrial System

In this chapter, the interactive dynamics of agriculture, industry and the ecosphere are modelled by a system of three ordinary differential equations using a predator-prey paradigm. It is however, assumed that the ecosphere is in a state of equilibrium, and hence the system reduces to a two dimensional one. We study the steady states and their local stabilities. We do this for various possible values of the parameters in the model. Criteria for global asymptotic stability for the steady states are also given. Numerical examples are also given for some parameter values. The numerical solutions are done with XPP.

Let A(t), I(t) and E(t) denote respectively, the agricultural assets, industrial assets and ecospheric assets (environmental quality) in whatever units they are measured. Agriculture generates its assets from the ecospheric assets. The process of generating the agricultural assets can be enhanced by industrial assets including machinery, pesticide, etc., at some cost. Depending on the asset creation by this process

and the cost, agricultural assets may increase, decrease or remain the same. Thus the interaction between agriculture and industry can be considered to be mutualism, parasitism or commensalism, since industry always generates its assets from agriculture. In general, we expect agriculture and industry to replenish the ecosphere. However, we will assume that the effect of industry on the ecosphere is negligible compared to that of agriculture. We also assume that industrial asset creation faces both fixed and variable expenses independent of agriculture and the ecosphere. Also, since agricultural asset creation depends on the quality of the environment, we allow the possibility of diminishing returns for agriculture.

Based on the above reasoning, we arrive at the following system of three ordinary differential equations to describe the interactions of the above three assets

$$\frac{dA}{dt} = \alpha EA - \beta A^2 + \gamma AI, \tag{2.1}$$

$$\frac{dI}{dt} = -\xi I - \eta I^2 + \delta AI, \tag{2.2}$$

$$\frac{dE}{dt} = -f(E, A) + g(E, A), \tag{2.3}$$

with initial conditions $A(0) \geq 0$, $I(0) \geq 0$, $E(0) \geq 0$, where α, β, ξ, η, δ are positive constants and γ is a negative, zero or positive constant depending on whether the interaction between agriculture and industry is parasitism, commensalism or mutualism. We also assume that the total ecospheric assets are bounded between zero and one, that is $0 \leq E(t) < 1$. The explicit form of f and g will be given in the next chapter, since in this chapter we are treating the ecospheric assets as constant. The constant α is the maximum growth rate of the agricultural assets, β is the diminishing returns coefficient for agriculture, γ represents the terms of trade between agriculture and industry, ξ and η denote respectively the constant and linear depreciation rates of industrial assets, and δAI represents the gain in assets by industry from agriculture.

As we stated earlier, in this chapter we assume that the ecosphere is in a state of equilibrium, that is, the uptake from the ecosphere by agriculture is equal to the input from agriculture, so E(t) is constant and hence $\frac{dE}{dt} = 0$. Therefore the system of equations in our model reduces to two, given by

$$\frac{dA}{dt} = \alpha_0 A - \beta A^2 + \gamma AI, \tag{2.4}$$

$$\frac{dI}{dt} = -\xi I - \eta I^2 + \delta AI, \tag{2.5}$$

where $A(0) \geq 0$, $I(0) \geq 0$ and $\alpha_0 = \alpha E$.

2.1 Local stability analysis of steady states

By applying the Hartman-Grobman theorem, we know that the local behavior of a nonlinear system near a hyperbolic fixed point can qualitatively be determined by the behavior of the corresponding linearized system near the origin. Thus the local stability of a hyperbolic fixed point of a nonlinear system can be determined from the Jacobian matrix J about the steady state. Hence the real parts of the eigenvalues of the Jacobian matrix J determine the local stability of a steady state ([6],[13],[21]). The Jacobian matrix $J(A, I)$ for the above system is given by

$$J(A, I) = \begin{bmatrix} \alpha_0 - 2\beta A + \gamma I & \gamma A \\ \delta I & -\xi - 2\eta I + \delta A \end{bmatrix}. \tag{2.6}$$

For steady states, $\frac{dA}{dt} = \frac{dI}{dt} = 0$, we have

$$\alpha_0 A - \beta A^2 + \gamma AI = 0, \tag{2.7}$$

$$-\xi I - \eta I^2 + \delta AI = 0. \tag{2.8}$$

2.2 Parasitism or Commensalism ($\gamma \leq 0$)

In this section, we solve Equations (2.7) and (2.8) for the required steady states using the fact that A(t)$\geq$0, I(t) ≥ 0 and

$$\beta\eta > \gamma\delta. \tag{2.9}$$

We obtain at least two steady states, given by $F_1 : (A, I) = (0, 0)$ and $F_2 : (A, I) = (\alpha_0/\beta, 0)$. There is the possibility of a third steady state given by $F_3 : (A, I) = (A^*, I^*) = (\frac{\eta\alpha_0 - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\alpha_0\delta - \beta\xi}{\beta\eta - \gamma\delta})$ provided

$$\alpha_0\delta > \beta\xi. \tag{2.10}$$

If F_3 exists, it is the only interior equilibrium or steady state for the system.

2.2.1 Local stability analysis of F_1

The Jacobian $J(A, I)$ evaluated at F_1 is given by

$$J_1 = J(0, 0) = \begin{bmatrix} \alpha_0 & 0 \\ 0 & -\xi \end{bmatrix}. \tag{2.11}$$

The eigenvalues of J_1 are $\alpha_0 > 0$ and $-\xi < 0$. Hence F_1 is a hyperbolic saddle point. F_1 is locally unstable in the A-direction and locally stable in the I-direction, that is if the interaction between agriculture and industry is parasitism or commensalism, then starting with a small industrial asset and a small agricultural asset, the agricultural asset will grow while the industrial asset will decline and become extinct.

2.2.2 Local stability analysis of F_2

The Jacobian $J(A, I)$ evaluated at F_2 is given by

$$J_2 = J(\alpha_0/\beta, 0) = \begin{bmatrix} -\alpha_0 & \frac{\gamma\alpha_0}{\beta} \\ 0 & \frac{-\xi\beta+\delta\alpha_0}{\beta} \end{bmatrix}. \tag{2.12}$$

The eigenvalues of J_2 are $\lambda_1 = -\alpha_0 < 0$ and $\lambda_2 = \frac{-\xi\beta+\delta\alpha_0}{\beta}$. If Equation (2.10) is satisfied (i.e. if F_3 exists) then $\lambda_2 > 0$, F_2 is a hyperbolic saddle point, which is locally stable in the A-direction and unstable in the I-direction. Thus with parasitism or commensalism between agriculture and industry, starting with a small industrial asset and agricultural asset near α_0/β, the industrial asset will grow while the agricultural asset may either grow, decline or remain the same. One would therefore expect such solutions to move toward to the interior equilibrium which is F_3. On the other hand, if $\alpha_0\delta < \beta\xi$ (i.e. if F_3 does not exist) then $\lambda_2 < 0$, and hence F_2 is locally stable. Thus starting with a small industrial asset and an agricultural asset near α_0/β, the industrial asset will decline and become extinct while the agricultural asset will turn to α_0/β. We show in Theorem 2.1 that F_2 in this case is not only a locally stable equilibrium but also is globally asymptotically stable if $4\beta\eta > (\gamma+\delta)^2$, that is, if F_3 does not exist then no matter the initial agricultural and industrial assets used, these assets will tend to F_2 in the long run if the parameters satisfy the above condition (see Figures 2.1 and 2.2).

2.2.3 Local stability analysis of F_3

If F_3 exists (i.e. if Equation (2.10) is satisfied), then the Jacobian $J(A, I)$ evaluated at F_3 is given by

$$J_3 = J(A^*, I^*) = \begin{bmatrix} \frac{\beta(\gamma\xi-\alpha_0\eta)}{\beta\eta-\gamma\delta} & \frac{\gamma(\eta\alpha_0-\gamma\xi)}{\beta\eta-\gamma\delta} \\ \frac{\delta(\alpha_0\delta-\beta\xi}{\beta\eta-\gamma\delta} & \frac{\eta(\beta\xi-\alpha_0\delta)}{\beta\eta-\gamma\delta} \end{bmatrix}. \tag{2.13}$$

Let the eigenvalues of J_3 be λ_1 and λ_2. Let τ and Δ denote the trace and the determinant of J_3 respectively. Then

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}, \tag{2.14}$$

where

$$\tau = \lambda_1 + \lambda_2 = \frac{\beta(\gamma\xi - \alpha_0\eta) + \eta(\beta\xi - \alpha_0\delta)}{\beta\eta - \gamma\delta} < 0,$$

and

$$\Delta = \lambda_1\lambda_2 = \frac{(\gamma\xi - \alpha_0\eta)(\beta\xi - \alpha_0\delta)}{\beta\eta - \gamma\delta} > 0,$$

since Equations (2.9) and (2.10) imply

$$\gamma\xi < \alpha_0\eta. \tag{2.15}$$

Thus F_3 (if it exists) is always locally stable because $\tau < 0$ and $\Delta > 0$ implies the real parts of λ_1 and λ_2 are negative. We show in Theorem 2.2 that F_3 is globally asymptotically stable if it exists and $4\beta\eta > (\gamma + \delta)^2$, that is if F_3 exists then no matter what the initial agricultural and Industrial assets, these assets will tend to F_3 provided $4\beta\eta > (\gamma + \delta)^2$ (see Figures 2.3 and 2.4).

2.2.4 Numerical examples

In this section we use XPP Software to study the phase portrait for our system for certain values of the parameters. We chose these values at random, just ensuring that

they satisfy our assumptions. The values of the parameters chosen do not represent any actual agricultural-industrial-ecospheric system:

$$\alpha_0 = 4.0, \quad \beta = 2.0, \quad \eta = 0.1, \quad \delta = 0.75.$$

The parameter ξ are chosen so as to either satisfy or violate Equation (2.10). Thus we choose ξ such that our system has two or three steady states. These will be considered as Case I and Case II. For each of the above cases, we study the phase portrait when γ= -1.0 and 0.0. These would represent the cases where we have parasitism and commensalism between agriculture and industry respectively.

CASE I (two steady states)

We set $\xi = 2.0$. One can verify that for each of the two values of γ chosen, $\alpha_0\delta < \beta\xi$, that is, our system has only two steady states given by $F_1 = (0,0)$ and $F_2 = (\alpha_0/\beta, 0)$.

EXAMPLE 2.1

$\gamma = -1.0$. Thus we assume agriculture will lose and industry will gain when they interact. The phase portrait for this scenario is shown in Figure 2.1.

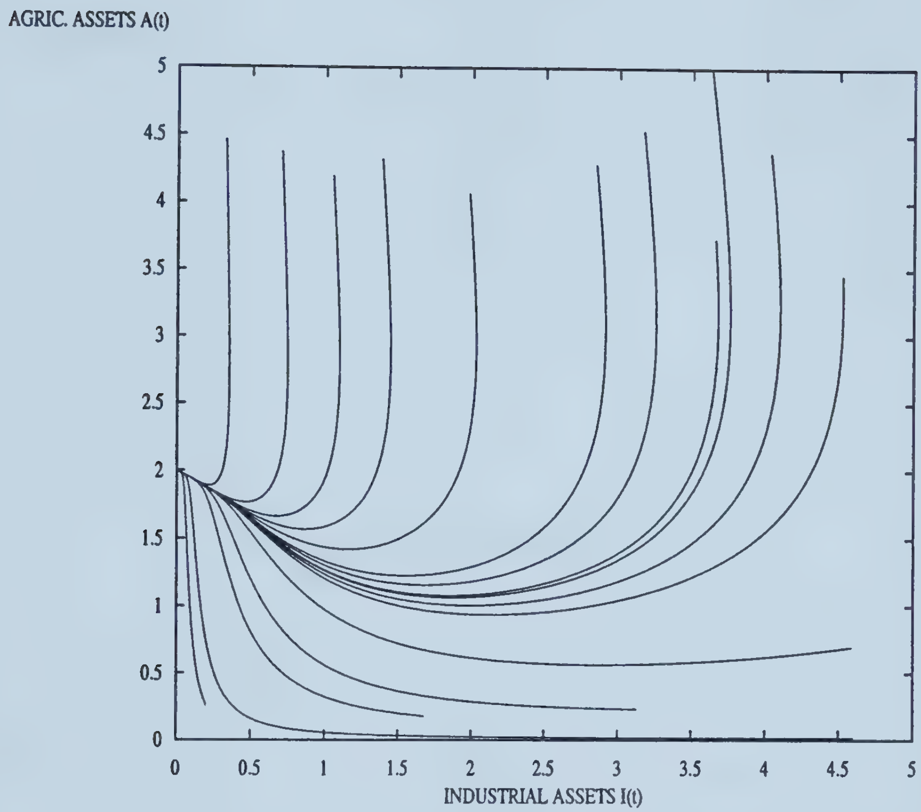


Figure 2.1: Phase portrait when $\gamma = -1.0$ and $\xi = 2.0$.

It can be seen from the phase portrait that, no matter where one starts, agricultural assets approach α_0/β while industrial assets become extinct. This suggests that F_2 is globally asymptotically stable for the values of the parameters chosen.

EXAMPLE 2.2

$\gamma = 0.0$. In this case we assume agricultural assets are unaffected when agriculture and industry interact but industry gains. The phase portrait for this example is as shown in Figure 2.2. Similar conclusions as in Example 2.1 can be drawn in this case.

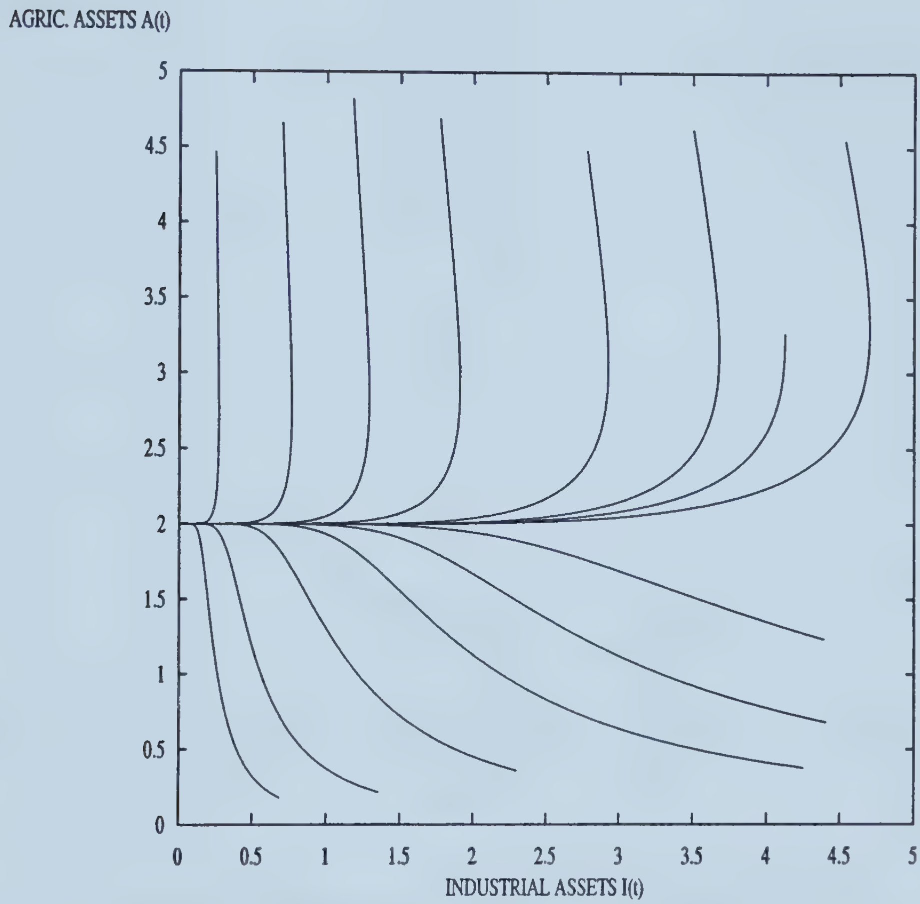


Figure 2.2: Phase portrait when $\gamma = 0.0$ and $\xi = 2.0$.

CASE II (three steady states)

In this case we set $\xi = 1.0$. It is easy to verify that $\alpha_0\delta > \beta\xi$. Thus our system will have three steady states namely F_1, F_2 and F_3 with F_3 being the only interior and only stable equilibrium.

EXAMPLE 2.3

In this case $\gamma = -1.0$. The phase portrait is as shown in Figure 2.3.

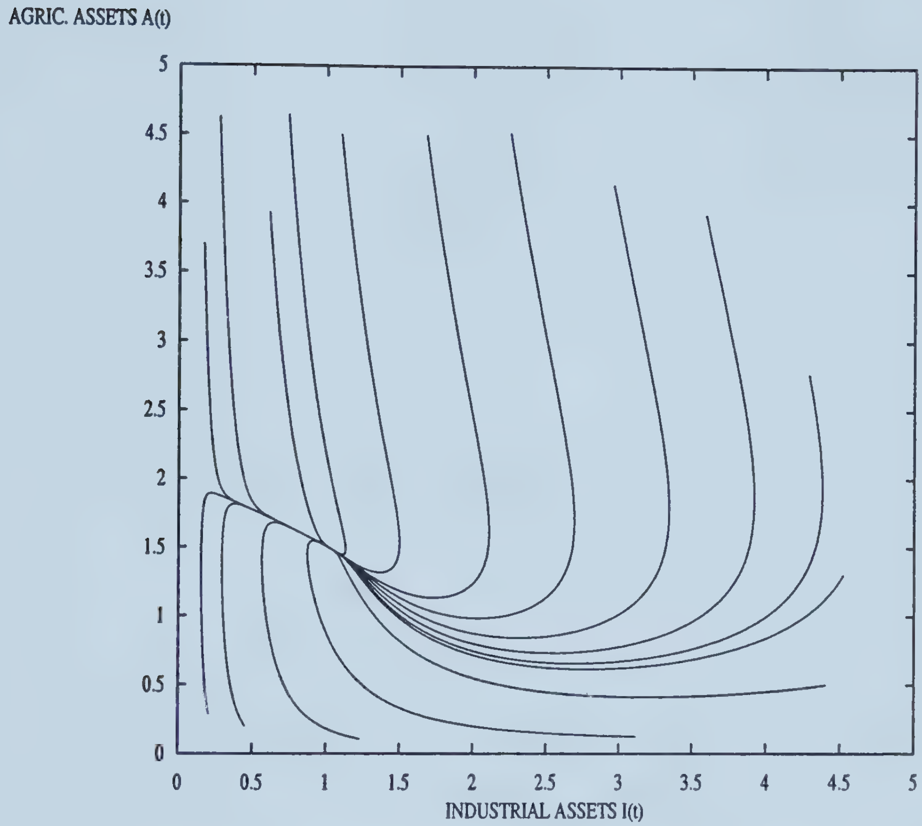


Figure 2.3: Phase portrait when $\gamma = -1.0$ and $\xi = 1.0$.

We see from the phase portrait that no matter what the initial condition on the assets, in the long run the assets tend to F_3, thus suggesting that F_3 is globally asymptotically stable in this case.

EXAMPLE 2.4

In this case $\gamma = 0.0$. The phase portrait obtained in this case is as shown in Figure 2.4. We observe that the phase portrait in Figures 2.3 and 2.4 are qualitatively the same. Hence the conclusions for Example 2.3 also apply to Example 2.4.

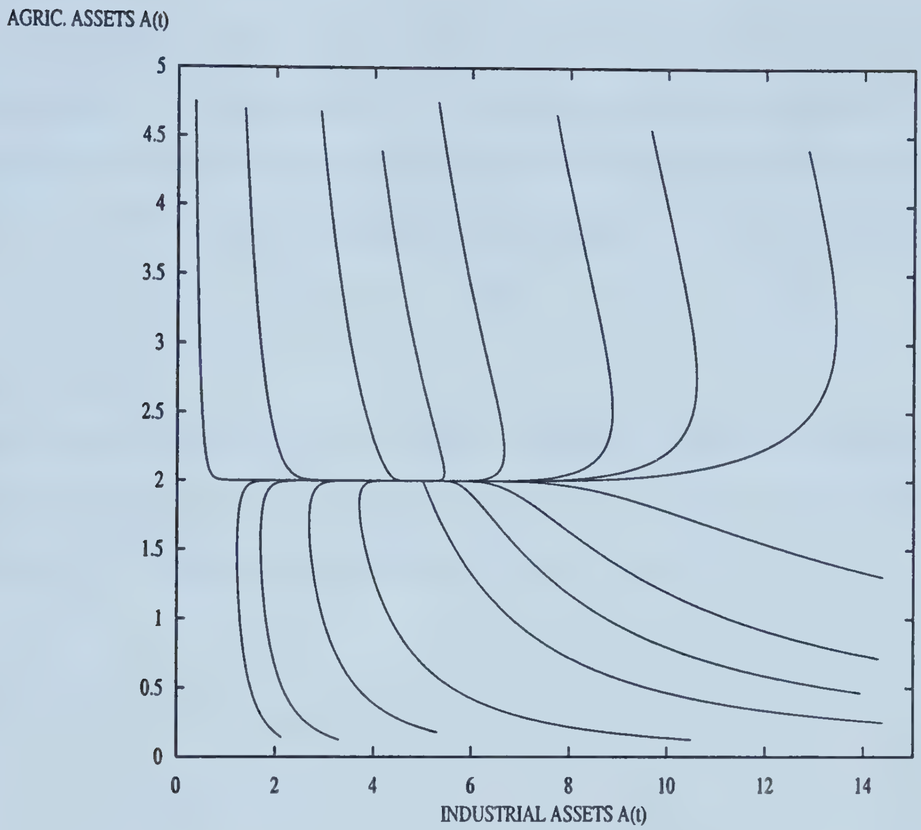


Figure 2.4: Phase portrait when $\gamma = 0.0$ and $\xi = 1.0$.

2.3 Mutualism ($\gamma > 0$)

In this section we study the steady states of Equations (2.4) and (2.5) with $\gamma > 0$. We do this under three different cases.

Case I: we choose γ such that $\beta\eta > \gamma\delta$. Case II: we choose γ such that $\beta\eta = \gamma\delta$. Case III: we choose γ such that $\beta\eta < \gamma\delta$.

In all these cases, we note that the steady states F_1 and F_2 always exist, but F_3 will only exist under different conditions in each case.

2.3.1 Case I : $\beta\eta > \gamma\delta$

In this case the third steady state F_3 exists only if Equation(2.10) is satisfied. Hence the analysis done in Section 2.2 for F_1, F_2, and F_3 holds in this case for F_1, F_2, and F_3 respectively. Also Theorems 2.1 and 2.2 hold in this case. We will as a result illustrate this case only with numerical examples. In Examples 2.5 and 2.6, we choose

$$\alpha_0 = 4.0, \quad \beta = 2.0, \quad \eta = 0.1, \quad \delta = 0.75, \quad \gamma = 0.2.$$

EXAMPLE 2.5

Let $\xi = 2.0$. These parameter values satisfy both Equation (2.9) and $\alpha_0\delta < \beta\xi$. Thus the system has only two steady states F_1 and F_2 with F_2 being locally asymptotically stable. The phase portrait is as shown in Figure 2.5.

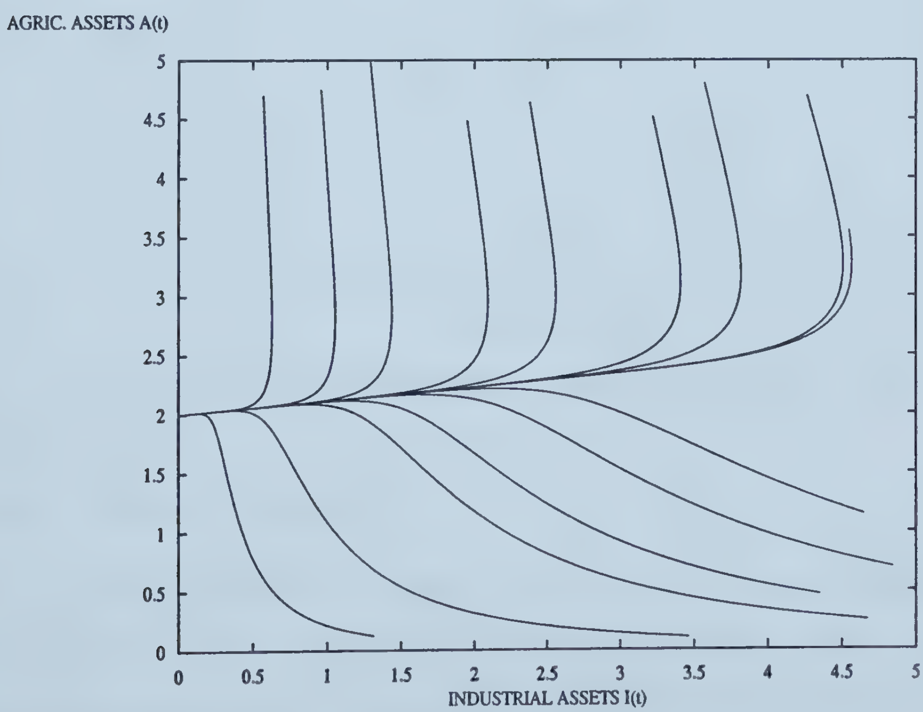


Figure 2.5: Phase portrait when $\gamma = 0.2$ and $\xi = 2.0$.

EXAMPLE 2.6

Let $\xi = 1.0$. Then the parameter values satisfy both Equations(2.9) and (2.10). Thus the system has steady states F_1, F_2 and F_3 with F_3 being locally asymptotically stable. The phase portrait is as shown in Figure 2.6.

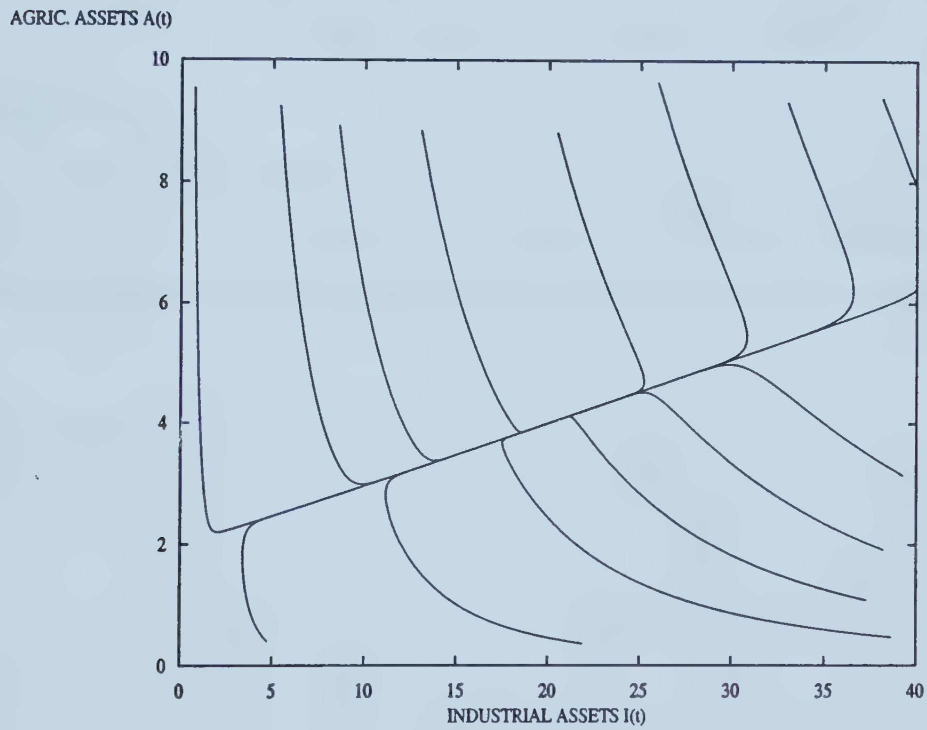


Figure 2.6: Phase portrait when $\gamma = 0.2$ and $\xi = 1.0$.

2.3.2 Case II: $\beta\eta = \gamma\delta$

Here the system has only two steady states F_1 and F_2. Using Equation (2.11), we obtain that F_1 is a hyperbolic saddle point which is locally unstable in the A-direction and locally stable in the I-direction. Using Equation (2.12), we obtain that F_2 is a hyperbolic saddle point if Equation (2.10) is satisfied. In that case F_2 is locally stable

in the A-direction and unstable in the I-direction. On the other hand if $\alpha_0\delta < \beta\xi$, then F_2 is locally asymptotically stable. One can show in this case, using similar arguments as in Theorem 2.1 that F_2 is globally asymptotically stable if it is locally asymptotically stable and $4\beta\eta > (\gamma+\delta)^2$. We illustrate the above with numerical examples. In Examples 2.7 and 2.8, we take

$$\alpha_0 = 4.0, \quad \beta = 2.0, \quad \eta = 0.1, \quad \delta = 0.75, \quad \gamma = 20/75.$$

EXAMPLE 2.7

Let $\xi = 2.0$, that is we have F_1 and F_2 as steady states with F_2 being locally asymptotically stable. The phase portrait is as shown in Figure 2.7. Here we observe that all solution curves or trajectories are bounded as in all the cases studied before.

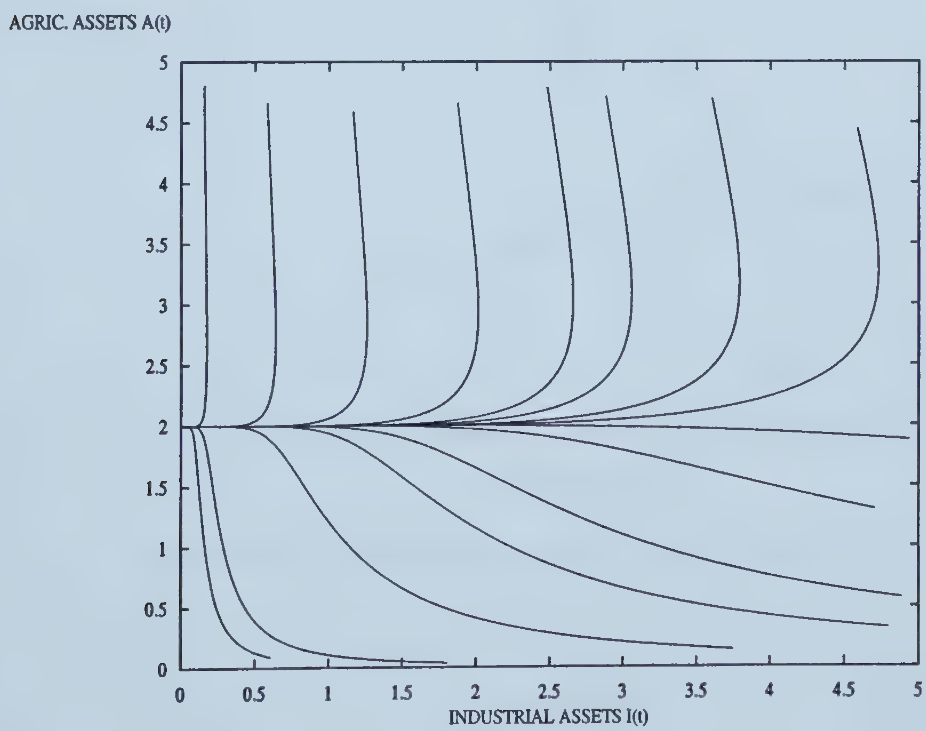


Figure 2.7: Phase portrait when $\gamma = 20/75$ and $\xi = 2.0$.

EXAMPLE 2.8

Let $\xi = 1.0$, that is we have F_1 and F_2 as steady states both of which are unstable. The phase portrait is as shown in Figure 2.8. In this case we observe that all trajectories are unbounded, that is no matter what the industrial assets and agricultural assets one starts with, both assets are going to grow unbounded in the long run.

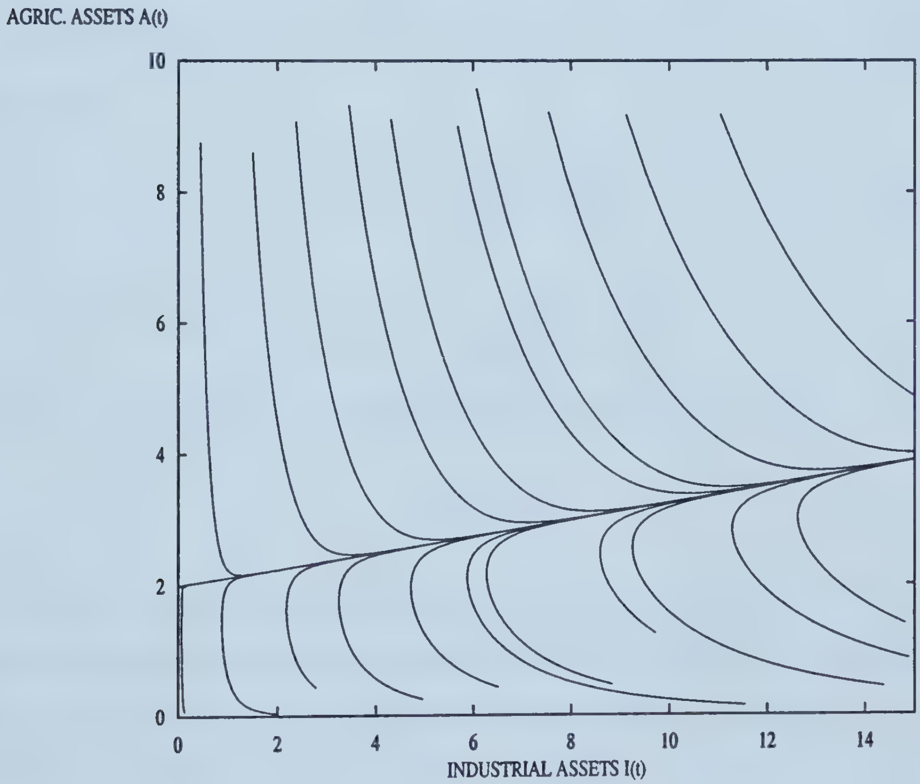


Figure 2.8: Phase portrait when $\gamma = 20/75$ and $\xi = 1.0$.

2.3.3 Case III: $\beta\eta < \gamma\delta$

The system always has two steady states F_1 and F_2. F_3 exists if

$$\alpha_0\delta < \beta\xi. \tag{2.16}$$

Using Equation (2.11), we obtain that F_1 is locally unstable as before. From Equation (2.12) we have F_2 is locally asymptotically stable if Equation (2.16) is satisfied (i.e. if F_3 exists) and is unstable if Equation (2.10) is satisfied. Also using Equation (2.13), we see that the trace and determinant of J_3, denoted by τ and Δ, are given respectively by

$$\tau = \lambda_1 + \lambda_2 = \frac{\beta(\gamma\xi - \alpha_0\eta) + \eta(\beta\xi - \alpha_0\delta)}{\beta\eta - \gamma\delta} < 0,$$

and

$$\Delta = \lambda_1\lambda_2 = \frac{(\gamma\xi - \alpha_0\eta)(\beta\xi - \alpha_0\delta)}{\beta\eta - \gamma\delta} < 0.$$

This follows from the fact that Equation(2.16) and $\beta\eta < \gamma\delta$ imply

$$\gamma\xi > \alpha_0\eta, \tag{2.17}$$

where λ_1 and λ_2 are the eigenvalues of J_3, that is the real part of one of the eigenvalues is negative while the real part of the other is positive. Hence F_3 is a hyperbolic saddle point. We illustrate the above with the following numerical examples. In Examples 2.9 and 2.10, we take

$$\alpha_0 = 4.0, \quad \beta = 2.0, \quad \eta = 0.1, \quad \delta = 0.75, \quad \gamma = 1.0.$$

EXAMPLE 2.9

Let $\xi = 2.0$. The phase portrait is as shown in Figure 2.9. It is clear from the portrait that depending on the initial agricultural and industrial asset values, trajectories could be bounded or unbounded.

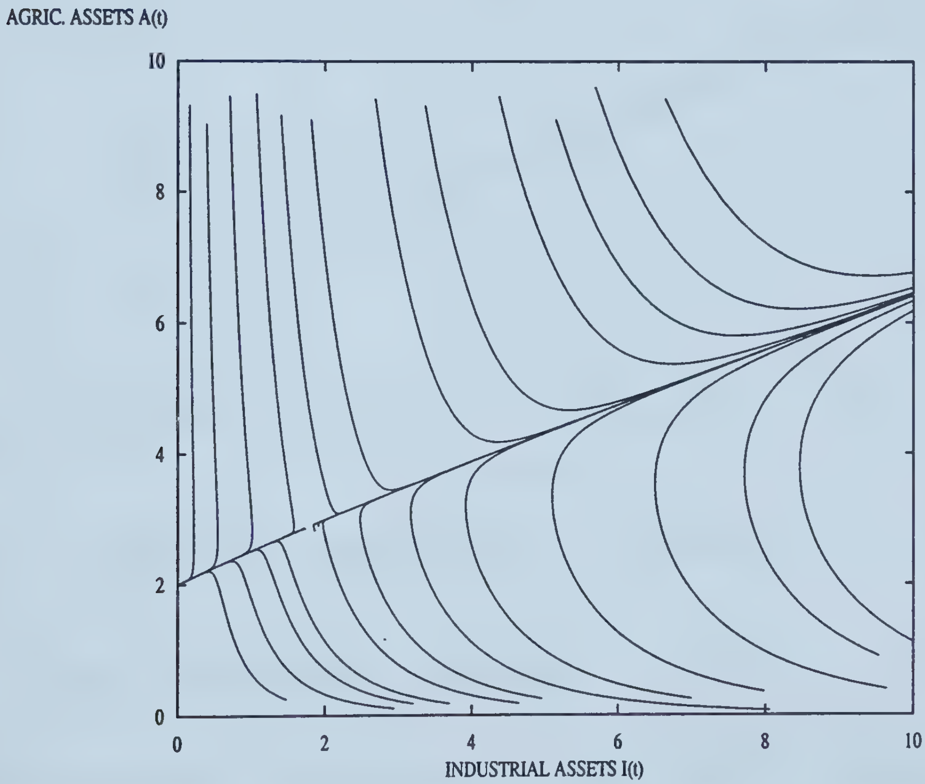


Figure 2.9: Phase portrait when $\gamma = 1.0$ and $\xi = 2.0$.

EXAMPLE 2.10

Let $\xi = 1.0$. Here we have F_1 and F_2 as steady states. The phase portrait is as shown in Figure 2.10. We see from the phase portrait that all trajectories are unbounded.

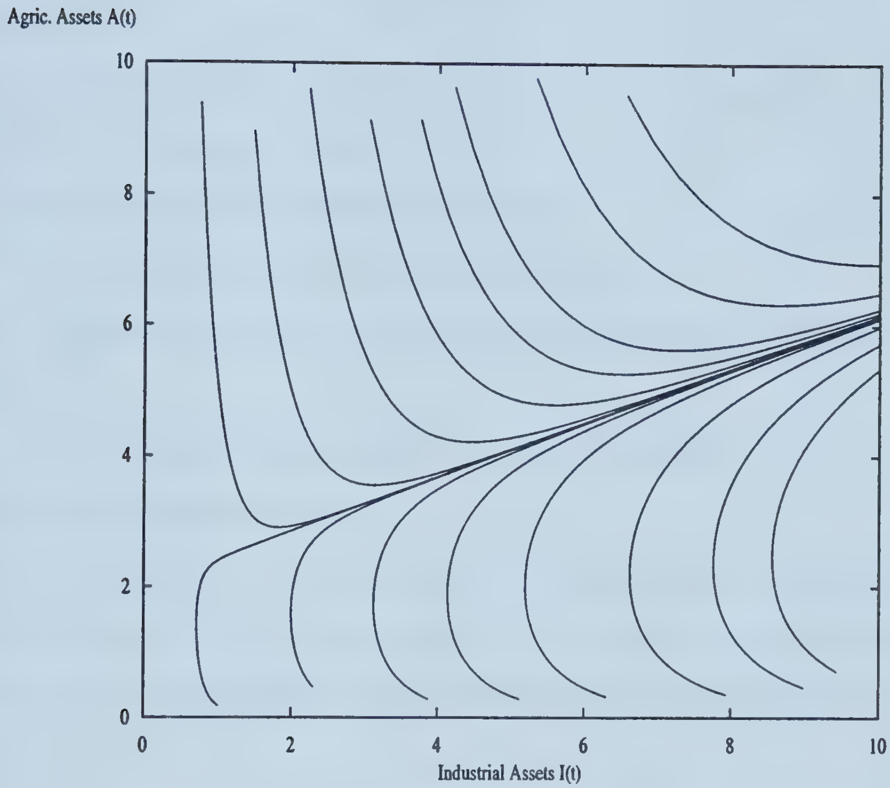


Figure 2.10: Phase portrait when $\gamma = 1.0$ and $\xi = 1.0$.

2.4 Global stability Analysis

In this section, we will establish criteria for the global asymptotic stability of F_2 and F_3. We recall that ([3],[11],[16]),

(a) a quadratic form $Q(X) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ is said to be negative definite if $Q(X) < 0$ for all $X \neq 0$ in $\Re^n$

(b) every quadratic form $Q(X) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ can be written in the form Q(X)=X^TAX where X is an $n \times 1$ vector in $\Re^n$, A is a symmetric $n \times n$ matrix, and $(.)^T$ denotes the transpose.

We make use of the lemmas in this section.

Lemma 2.1

Let A be a symmetric $n \times n$ matrix, X be an $n \times 1$ vector and $V = X^T AX$, where $(.)^T$ denotes the transpose. Then

(i) V is negative if $X^T AX$ is negative definite,

(ii) $X^T AX$ is negative definite if A is negative definite,

(iii) A is negative definite if the eigenvalues of A have negative real parts.

Proof

The proof of the above lemma can be found in [8] and [17]].

Lemma 2.2 [Frobenius 1876]

Let A be a symmetric $n \times n$ matrix over the reals. Then a necessary and sufficient condition for the real, symmetric matrix A to be negative definite is that the principal minors of A, starting with that of the first order, be alternatively negative and positive.

Proof

The proof of the above Lemma can be found in [8].

Lemma 2.3 [Liapunov]

Let $\frac{dx}{dt} = f(x)$ be a nonlinear system in $\Re^n$. Let G be an open subset of $\Re^n$ containing x_0. Suppose that f$\in$ $C'(G)$ and that f(x_0)=0. Suppose further that there exists a function V$\in$ $C'(G)$ satisfying V(x_0)=0 and $V(x) > 0$ if $x \neq x_0$. Then

(i) if $\frac{d}{dt}V(x) \leq 0$ for all $x \in G$, x_0 is stable,

(ii) if $\frac{d}{dt}V(x) < 0$ for all $x \in G\backslash\{x_0\}$, x_0 is asymptotically stable,

(iii) if $\frac{d}{dt}V(x) > 0$ for all $x \in G\backslash\{x_0\}$, x_0 is unstable.

Proof

The proof can be found in [21].

Lemma 2.4

Let ρ be a positive real number. Then for $x > 0$, the function $y = x - \rho - \rho \ln(x/\rho) > 0$ if $x \neq \rho$.

Proof

Clearly y=0 if $x = \rho$. Also $\frac{dy}{dx} = 1 - (\rho/x)$, that is, if $x > 0$ then the function y is strictly decreasing if $x < \rho$ and is strictly increasing if $x > \rho$. Hence $x = \rho$ is an absolute minimum of the function y for $x > 0$.

Theorem 2.1

The steady state $F_2 = (\alpha_0/\beta, 0)$ is globally asymptotically stable if the following conditions are satisfied

(i) $\beta\eta \geq \gamma\delta$.

(ii) $4\beta\eta > (\gamma + \delta)^2$.

(iii) $\alpha_0\delta < \beta\xi$.

Proof

Let $V(A, I) = A - \alpha_0/\beta - \alpha_0/\beta \ln(\beta A/\alpha_0) + I$. Then by Lemma 2.4, $V(A, I) > 0$ if $(A, I) \neq (\alpha_0/\beta, 0) = F_2$ since $A \geq 0$ and $I \geq 0$. Also $V(A, I) = 0$ if $(A, I) = (\alpha_0/\beta, 0) = F_2$. Also

$$\begin{aligned}
\frac{dV(A,I)}{dt} &= \frac{dA}{dt} - \left(\frac{\alpha_0}{\beta A}\right)\frac{dA}{dt} + \frac{dI}{dt} \\
&= 2\alpha_0 A - \beta A^2 + (\gamma+\delta)^2 - \eta I^2 - (\xi + \alpha_0\gamma/\beta)I - \alpha_0^2/\beta \quad \text{by Eqn (2.4) and (2.5} \\
&= -\beta(A - \alpha_0/\beta)^2 - \eta I^2 + (\gamma+\delta)(A - \alpha_0/\beta)I + (\delta\alpha_0 - \beta\xi)I/\beta \\
&\leq -\beta(A - \alpha_0/\beta)^2 - \eta I^2 + (\gamma+\delta)(A - \alpha_0/\beta)I \\
&= X^T B X,
\end{aligned}$$

where $X^T = (A - \alpha_0/\beta \quad I)$ and $B = \begin{bmatrix} -\beta & \frac{\gamma+\delta}{2} \\ \frac{\gamma+\delta}{2} & -\eta \end{bmatrix}$.

Let τ and Δ denote respectively the trace and determinant of B. Let λ_1 and λ_2 denote the eigenvalues of B. Then

$$\tau = \lambda_1 + \lambda_2 = -(\beta + \eta) < 0,$$

$$4\Delta = 4\beta\eta - (\gamma + \delta)^2 > 0 \qquad \text{by} \quad \text{condition} \quad \text{(ii)}.$$

Hence both eigenvalues of B have negative real parts. Then by Lemma 2.2, B is negative definite. Also by Lemma 2.1 and Lemma 2.2 $X^T BX$ is negative definite. Hence $\frac{dV(A,I)}{dt} < 0$ if $(A, I) \neq (\alpha_0/\beta, 0) = F_2$. Thus, by Lemma 2.3 F_2 is globally asymptotically stable.

Theorem 2.2

The steady state $F_3 = (A^*, I^*) = (\frac{\eta\alpha_0 - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\alpha_0\delta - \beta\xi}{\beta\eta - \gamma\delta})$ is globally asymptotically stable if

(i) $\beta\eta > \gamma\delta$.

(ii) $4\beta\eta > (\gamma + \delta)^2$.

(iii) $\alpha_0\delta > \beta\xi$.

Proof

Let $V(A, I) = A - A^* - A^* ln(A/A^*) + I - I^* - I^* \ln(I/I^*)$. As shown before, $V(A, I) > 0$ if $(A, I) \neq (A^*, I^*) = (\frac{\eta\alpha_0 - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\alpha_0\delta - \beta\xi}{\beta\eta - \gamma\delta})$ and $V(A, I) = 0$ if $(A, I) = (A^*, I^*)$.

Also

$$
\begin{aligned}
\frac{dV(A,I)}{dt} &= \frac{dA}{dt} - \frac{A^*}{A}\frac{dA}{dt} + \frac{dI}{dt} - \frac{I^*}{I}\frac{dI}{dt} \\
&= (\frac{A-A^*}{A})(\alpha_0 A + \gamma AI - \beta A^2) + (\frac{I-I^*}{I})(-\xi I + \delta AI - \eta I^2) \\
&= (A-A^*)(\alpha_0 + \gamma I - \beta A) + (I-I^*)(-\xi + \delta A^* - \eta I^*) \\
&= (A-A^*)(\alpha_0 + \gamma I^* - \beta A^*) + (I-I^*)(-\xi + \delta A - \eta I) - \beta(A-A^*)^2 - \eta(I-I^*)^2 \\
&\quad\quad + (\gamma+\delta)(A-A^*)(I-I^*) \\
&= -\beta(A-A^*)^2 - \eta(I-I^*)^2 + (\gamma+\delta)(A-A^*)(I-I^*) \qquad (***) \\
&< 0 \qquad if \quad (A,I) \neq (A^*,I^*) \quad \text{(see proof of Theorem 2.1).}
\end{aligned}
$$

Hence by Lemma 2.3, $F_3 = (A^*, I^*)$ is globally asymptotically stable.

JUSTIFICATION OF (***)

$$\alpha_0 + \gamma I^* - \beta A^* = \alpha_0 + \gamma(\frac{\alpha_0\delta - \beta\xi}{\beta\eta - \gamma\delta}) - \beta(\frac{\eta\alpha_0 - \gamma\xi}{\beta\eta - \gamma\delta}) = 0$$

and

$$-\xi + \delta A^* - \eta I^* = -\xi + \delta(\frac{\eta\alpha_0 - \gamma\xi}{\beta\eta - \gamma\delta}) - \eta(\frac{\alpha_0\delta - \beta\xi}{\beta\eta - \gamma\delta}) = 0.$$

Chapter 3

Ecospheric Recovery Model of an Agricultural-Industrial System

In the previous chapter, we studied the agricultural-industrial system by assuming that the ecospheric assets were in equilibrium, which is commonly the practice for most subsistence farmers in developing countries. In this chapter, we study the practice of most non-subsistence farmers in both developing and developed countries. The farmers we are considering are those who assume that there will be a loss in ecospheric assets as a result of their agricultural activity and hence there is a need to correct that loss in ecospheric assets in order to increase both their agricultural and industrial assets. However, because they perform in every given period the same agricultural activity on the ecosphere, the amount of environmental quality such as nutrients they add to the ecosphere to correct this loss in ecospheric assets at any given period is the same. We assume that because of the cost of adding environmental quality to the ecosphere assets, no farmer will incur this cost if he or she is not performing any agricultural activity on the ecosphere. We also assume that no matter what the initial

state of the ecosphere is, in the absence of any agricultural activity the ecosphere is naturally going to recover and become richer in the long run. Thus, in this chapter, we investigate the consequences of introducing the ecospheric equation based on the above reasoning into our agricultural-industrial model by studying the steady states for the system and their stabilities. We also give some numerical examples. The numerics are done with XPP software.

3.1 The Model

Let A(t), I(t) and E(t) denote respectively, the agricultural assets, industrial assets and ecospheric assets. Then based on the above reasoning together with that from the previous chapter, we arrive at the following system of three ordinary differential equations to describe the interaction of the assets

$$\frac{dA}{dt} = \alpha EA - \beta A^2 + \gamma AI, \tag{3.1}$$

$$\frac{dI}{dt} = -\xi I - \eta I^2 + \delta AI, \tag{3.2}$$

$$\frac{dE}{dt} = -\kappa EA + \vartheta(1-E)E + \mu, \tag{3.3}$$

with initial conditions $A(0) \geq 0, I(0) \geq 0, E(0) \geq 0$, where $0 \leq \mu < 1$, $0 \leq E(t) \leq 1$, κ and ϑ are positive constants. We note that in order for $0 \leq E(t) \leq 1$ to be statisfied, we require

$$\mu \leq \kappa < \vartheta. \tag{3.4}$$

Here, κ is the maximum rate of degradation of the ecosphere due to agriculture, ϑ is the natural restoration (recovery) rate of the ecosphere and μ is the constant correction rate for the ecosphere due to the performance of agriculture activity. The other parameters are as defined in Chapter 1.

3.2 Steady states

The steady state conditions for Equations (3.1), (3.2) and (3.3) are found as solutions to the system $\frac{dA}{dt} = \frac{dI}{dt} = \frac{dE}{dt} = 0$, that is, solutions to the system

$$\alpha EA - \beta A^2 + \gamma AI = 0, \tag{3.5}$$

$$-\xi I - \eta I^2 + \delta AI = 0, \tag{3.6}$$

$$-\kappa EA + \vartheta(1 - E)E + \mu = 0. \tag{3.7}$$

We are only interested in nonnegative steady states with $0 \le E(t) \le 1$. We will group these steady states into (boundary) axial, (boundary)planar and interior steady states. It is obvious from Equation (3.7) that $E(t) \neq 0$ if $\mu \neq 0$. Hence there is no steady state in the (A,I)-plane if $\mu \neq 0$.

3.2.1 Axial steady states

Consider the case when $A = I = 0$. Then we obtain from Equation (3.7) that

$$E = \frac{1}{2}(1 \pm \sqrt{\frac{4\mu + \vartheta}{\vartheta}}). \tag{3.8}$$

If $\mu = 0$ then E= 0 or 1. If $\mu \neq 0$, then there will be no permissible steady state, since E will be negative or larger than one. This is consistent with our initial assumption that if there is no agricultural activity on the ecosphere, then $\mu = 0$. We denote these steady states (i.e. for $\mu = 0$) by $F_{1A} = (0, 0, 0)$ and $F_{2A} = (0, 0, 1)$. These are the only axial steady states since $E \neq 0$ if $\mu \neq 0$.

3.2.2 Non-axial planar steady states

Here we examine the planar steady states of our system under two separate cases; (I) $\mu = 0$ and (II) $\mu \neq 0$.

Case I: $\mu = 0$

In this case Equations (3.5), (3.6) and (3.7) become

$$\alpha EA - \beta A^2 + \gamma AI = 0, \tag{3.9}$$

$$-\xi I - \eta I^2 + \delta AI = 0, \tag{3.10}$$

$$-\kappa EA + \vartheta(1 - E)E = 0. \tag{3.11}$$

In the interior of the (A,I)-plane, we have $E = 0$, and hence we obtain from Equation (3.9) that $I = \beta A/\gamma$ which is positive only if $\gamma > 0$. Now assuming that $\gamma > 0$ and substituting for I in Equation (3.10), we obtain $A = \frac{\xi\gamma}{\gamma\delta - \beta\eta}$. We note that $A > 0$ only if $\gamma\delta > \beta\eta$. Thus if we assume that $\gamma > 0$ and $\gamma\delta > \beta\eta$, then there is a steady state in the (A,I)-plane. Denote this steady state by $F_{1P} = (\frac{\xi\gamma}{\gamma\delta-\beta\eta}, \frac{\xi\beta}{\gamma\delta-\beta\eta}, 0)$.

In the interior of the (A,E)-plane, we have $I = 0$, and hence we obtain from Equation (3.9) that $A = \alpha E/\beta$. Substituting for A in Equation (3.11) and solving for E, we obtain $E = \frac{\vartheta\beta}{\kappa\alpha+\vartheta\beta}$. Thus we always have a steady state in the (A,E)-plane. Denote this by $F_{2P} = (\frac{\vartheta\alpha}{\kappa\alpha+\vartheta\beta}, 0, \frac{\vartheta\beta}{\kappa\alpha+\vartheta\beta})$.

There is no steady state in the interior of the (I,E)-plane. This is because in the (I,E)-plane $A = 0$, and hence we obtain from Equation (3.10) that either $I = 0$ or I is negative, which is not feasible.

Case II: $\mu \neq 0$

There is no steady state in the interior of the (A,I)-plane because $E \neq 0$ if $\mu \neq 0$. Also there is no steady state in the interior of the (I,E)-plane, since if $A = 0$ then by Equation (3.6) either $I = 0$ or I is negative which is not feasible. In the interior of the (A,E)-plane, $I = 0$ and $A \neq 0$, and hence Equation (3.5) reduces to

$$A = \frac{\alpha E}{\beta}. \tag{3.12}$$

We substitute for A in Equation (3.7) and solve for E to get

$$E = E^* = \frac{\beta\vartheta + \sqrt{\beta^2\vartheta^2 + 4\mu\beta(\beta\vartheta + \kappa\alpha)}}{2(\beta\vartheta + \kappa\alpha)}. \tag{3.13}$$

It is easy to verify that $0 < E^* \leq 1$ if

$$\mu\beta \leq \kappa\alpha. \tag{3.14}$$

Equation (3.14) is always satisfied since the coefficient of diminishing returns for agriculture β is assumed to be less than the maximum growth rate of agriculture α and also the constant rate of correction for the ecosphere μ is less than the maximum rate of degradation of the ecosphere κ. Hence E^* always exists. Denote the corrsponding steady state by $F_{3P} = (\frac{\alpha E^*}{\beta}, 0, E^*)$, where E^* is given by Equation (3.13).

3.2.3 Interior steady states

Here we solve Equations (3.5), (3.6) and (3.7) assuming $A > 0$, $I > 0$ and $0 < E \leq 1$. Then since $A > 0$ and $I > 0$, we may divide Equations (3.5) and (3.6) by A and I respectively to obtain the following system

$$\alpha E - \beta A + \gamma I = 0, \tag{3.15}$$

$$-\xi - \eta I + \delta A = 0, \tag{3.16}$$

$$-\kappa E A + \vartheta(1 - E)E + \mu = 0. \tag{3.17}$$

At this point, we break our analysis into two cases; (I) $\mu = 0$, (II) $\mu > 0$.

Case I : $\mu = 0$

We break this case into three subcases; (a) $\gamma = 0$, (b) $\gamma < 0$, and (c) $\gamma > 0$.

Subcase Ia: $\gamma = 0$, $\mu = 0$

Here we solve Equation (3.15), to get

$$E = \frac{\beta A}{\alpha}. \tag{3.18}$$

Substituting Equation (3.18) into (3.17) and solving for A, we obtain

$$A = \frac{\alpha\vartheta}{\alpha\kappa + \beta\vartheta}. \tag{3.19}$$

Substituting Equation (3.19) into (3.16) and solving for I, we obtain

$$I = \frac{\delta\alpha\vartheta - \alpha\kappa\xi - \beta\vartheta\xi}{\eta\alpha\kappa + \eta\beta\vartheta}. \tag{3.20}$$

We observe that $I > 0$ only if

$$\delta\alpha\vartheta > \alpha\kappa\xi + \beta\vartheta\xi. \tag{3.21}$$

Now assuming Equation (3.21) is satisfied, then we have an interior steady state for our system if $\gamma = \mu = 0$. Denote this steady state by $F_{1I} = (\frac{\alpha\vartheta}{\alpha\kappa+\beta\vartheta}, \frac{\delta\alpha\vartheta-\alpha\kappa\xi-\beta\vartheta\xi}{\eta\alpha\kappa+\eta\beta\vartheta}, \frac{\beta\vartheta}{\alpha\kappa+\beta\vartheta})$.

Subcase Ib: $\gamma < 0$, $\mu = 0$

We solve Equations (3.15), (3.16), and (3.17) under the above conditions. We obtain from Equation (3.17) that

$$E = \frac{\vartheta - \kappa A}{\vartheta}. \tag{3.22}$$

We substitute (3.22) into (3.15) and solve for I to get

$$I = \frac{(\alpha\kappa + \beta\vartheta)A - \alpha\vartheta}{\gamma\vartheta}. \tag{3.23}$$

Now substitute (3.13) into (3.16) to get

$$A = \frac{\vartheta(\alpha\eta - \gamma\xi)}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}. \tag{3.24}$$

Finally, substitute (3.24) into (3.23) and (3.22) to get respectively

$$I = \frac{\alpha\vartheta\delta - \xi(\alpha\kappa + \beta\vartheta)}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)} \tag{3.25}$$

and

$$E = \frac{\beta\eta\vartheta - \gamma\delta\vartheta + \kappa\gamma\xi}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}. \tag{3.26}$$

We observe from (3.25) that if $\alpha\delta\vartheta > \xi(\alpha\kappa+\beta\vartheta)$ then $I > 0$, but if $\alpha\delta\vartheta \le \xi(\alpha\kappa+\beta\vartheta)$ then $I \le 0$. Thus if (3.21) is satisfied then $I > 0$. It is very easy to verify that if Equation (3.21) is satisfied, then from (3.26) and (3.23) we have $0 < E < 1$. Hence our system has an interior steady state for $\gamma < 0$ and $\mu = 0$ given by

$$F_{2I} = \left(\frac{\vartheta(\alpha\eta - \gamma\xi)}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}, \frac{\alpha\vartheta\delta - \xi(\alpha\kappa + \beta\vartheta)}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}, \frac{\beta\eta\vartheta - \gamma\delta\vartheta + \kappa\gamma\xi}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}\right),$$

provided $\alpha\delta\vartheta > \xi(\alpha\kappa + \beta\vartheta)$.

Subcase Ic: $\gamma > 0$, $\mu = 0$

Here we solve Equations (3.15),(3.16) and (3.17) for A, I and E under the above conditions. We obtain

$$A = \frac{\vartheta(\alpha\eta - \gamma\xi)}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}, \qquad I = \frac{\alpha\vartheta\delta - \xi(\alpha\kappa + \beta\vartheta)}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}, \qquad E = \frac{\beta\eta\vartheta - \gamma\delta\vartheta + \kappa\gamma\xi}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)},$$

which are Equations (3.24), (3.25) and (3.26) respectively. The above gives a feasible steady state (i.e $0 < E < 1$, $A > 0$, $I > 0$) if either of the following conditions is satisfied.

$$(i) \qquad \frac{\alpha\vartheta\delta}{\xi} < \alpha\kappa + \beta\vartheta < \frac{\vartheta\gamma\delta}{\eta}.$$

$$(ii) \qquad \frac{\alpha\vartheta\delta}{\xi} > \alpha\kappa + \beta\vartheta > \frac{\vartheta\gamma\delta}{\eta}.$$

Since by our assumption, η is very small, it is more likely that the first condition will normally be satisfied instead of the second. The second condition can only occur if

γ is small. In the later case we have an interior steady state if agriculture generates little or no assets from industry. Thus for $\gamma > 0$, $\mu = 0$, we have an internal steady state for our system given by

$$F_{2I} = \left(\frac{\vartheta(\alpha\eta - \gamma\xi)}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}, \frac{\alpha\vartheta\delta - \xi(\alpha\kappa + \beta\vartheta)}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}, \frac{\beta\eta\vartheta - \gamma\delta\vartheta + \kappa\gamma\xi}{\alpha\eta\kappa + \vartheta(\beta\eta - \gamma\delta)}\right),$$

provided condition (i) or (ii) is satisfied.

Case II: $\mu > 0$

Here we solve the system of Equations given by (3.15), (3.16) and (3.17) for steady states assuming that $\mu > 0$. We do this by considering three subcases; (a) $\gamma = 0$, (b) $\gamma < 0$ and (c) $\gamma > 0$.

Subcase IIa: $\gamma = 0, \quad \mu > 0$

From Equation (3.15), we obtain $A = \frac{\alpha E}{\beta}$. Substitute this in (3.17) to get

$$E = E_{1I}^{*} = \frac{\beta\vartheta + \sqrt{\beta^2\vartheta^2 + 4\mu\beta(\beta\vartheta + \kappa\alpha)}}{2(\beta\vartheta + \kappa\alpha)}. \tag{3.27}$$

Clearly, $0 < E_{1I}^{*} \leq 1$, (by Equations (3.13) and (3.14)). Substituting E_{1I}^{*} into Equation (3.16), we obtain

$$I = \frac{-\xi + \delta A}{\eta} = \frac{-\beta\xi + \delta\alpha E_{1I}^{*}}{\eta\beta}, \tag{3.28}$$

which exists if $\frac{\beta\xi}{\delta\alpha} < E_{1I}^{*}$, which we now assume. Denote this steady state by

$$F_{3I} = (\alpha E_{1I}^{*}/\beta, (-\beta\xi + \delta\alpha E_{1I}^{*})/\eta\beta, E_{1I}^{*}),$$

where E_{1I}^{*} is given by Equation (3.27).

Subcase IIb: $\gamma < 0, \quad \mu > 0$

We solve Equation (3.16) for I and obtain

$$I = \frac{\delta A - \xi}{\eta}. \tag{3.29}$$

Note that equation (3.29) shows that if $I > 0$ then $A > 0$. Substitute (3.29) into (3.15) and solve for A to get

$$A = \frac{\alpha\eta E - \gamma\xi}{\beta\eta - \gamma\delta}. \tag{3.30}$$

Substitute (3.30) into (3.14) and solve for E to get

$$E = E_{2I}^{\pm} = \frac{(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi \pm \sqrt{[(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi]^2 + 4\mu(\beta\eta - \gamma\delta)[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha]}}{2[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha]}. \tag{3.31}$$

Since $\gamma < 0$ and $\mu > 0$, $\beta\eta - \gamma\delta > 0$, and hence for a feasible steady state (i.e. $E > 0$) we have $E = E_{2I}^{+}$. Now substitute the above expression for E into Equation (3.30) and then into Equation (3.29) to get

$$I = \frac{\delta\alpha E_{2I}^{+} - \beta\xi}{\beta\eta - \gamma\delta}, \tag{3.32}$$

which exists (i.e. I is positive) if

$$\frac{\beta\xi}{\delta\alpha} < E_{2I}^{+} \leq 1. \tag{3.33}$$

Now assuming this relation is satisfied, then we have an interior steady state. Denote this by

$$F_{4I}^{+} = \left(\frac{\alpha\eta E_{2I}^{+} - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\delta\alpha E_{2I}^{+} - \beta\xi}{\beta\eta - \gamma\delta}, E_{2I}^{+}\right),$$

where E_{2I}^{+} is given by equation (3.31).

Subcase IIc: $\gamma > 0, \quad \mu > 0$

Here we solve Equations (3.15), (3.16) and (3.17) for A, I and E. We obtain as before,

$$E = E_{2I}^{\pm}, I = \frac{\delta\alpha E_{2I}^{\pm} - \beta\xi}{\beta\eta - \gamma\delta}, A = \frac{\alpha\eta E_{2I}^{\pm} - \gamma\xi}{\beta\eta - \gamma\delta},$$

where $E_{2I}^{\pm}$ is given by Equation (3.31). In this case, if γ is choosen such that $\beta\eta - \gamma\delta > 0$ then we have an interior steady state given by F_{4I}^{+}. Thus we have an

interior steady state given by F_{4I}^{+} if

$$\beta\eta > \gamma\delta \quad \text{and} \quad \frac{\beta\xi}{\alpha\delta} < E_{2I}^{+} \leq 1.$$

If on the other hand γ satisfies the relation $\beta\eta < \gamma\delta$, then from (3.31) there may be no, one, or two feasible values for E (i.e. real positive values of E) depending on the choice of the values of the other parameters. Thus in this case we may have zero, one or two possible interior equilibria. The various cases are as follows:

(i) If $(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha < 0$, then $E = E_{2I}^{-}$. In this case we will have a steady state given by $F_{4I}^{-} = (\frac{\alpha\eta E_{2I}^{-} - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\delta\alpha E_{2I}^{-} - \beta\xi}{\beta\eta - \gamma\delta}, E_{2I}^{-})$ provided

$$0 < E_{2I}^{-} < \frac{\beta\xi}{\alpha\delta} \quad \text{and} \quad \beta\eta < \gamma\delta.$$

(ii) If

$$(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha > 0, \quad (\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi > 0 \quad \text{and}$$

$$[(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi]^2 + 4\mu(\beta\eta - \gamma\delta)[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha] \geq 0,$$

then $E = E_{2I}^{\pm}$. Thus in this case we have two steady states given by $F_{4I}^{\pm}(A, I, E) = (\frac{\alpha\eta E_{2I}^{\pm} - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\delta\alpha E_{2I}^{\pm} - \beta\xi}{\beta\eta - \gamma\delta}, E_{2I}^{\pm})$ provided all the above conditions are satisfied together with the following

$$\beta\eta < \gamma\delta \quad \text{and} \quad 0 < E_{2I}^{\pm} < \frac{\beta\xi}{\alpha\delta}.$$

We note that if

$$[(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi]^2 + 4\mu(\beta\eta - \gamma\delta)[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha] = 0,$$

then F_{4I}^{+} coinsides with F_{4I}^{-}.

In the $\beta\eta > \gamma\delta$ case, we have an interior steady state occurring at a higher ecospheric asset level, while in the $\beta\eta < \gamma\delta$ case, at a lower ecospheric asset level. This is not

surprising, because the first condition can only be satisfied if γ is small or negative (i.e. the terms of trade favour industry more than agriculture), and hence by Equation (3.1), there will be less growth in agriculture, which in turn lowers the growth rate in industry and the degradation rate in the ecosphere. The reverse scenario occurs if the second condition is satisfied.

In general, one would not expect $\beta\eta = \gamma\delta$. But if this occurs, then solving Equations (3.15), (3.16) and (3.17) we get

$$E = \frac{\gamma\xi}{\alpha\eta} = \frac{\beta\xi}{\alpha\delta}, \tag{3.34}$$

$$A = \frac{\mu(\alpha\eta)^2 + \vartheta(\alpha\eta - \gamma\xi)\gamma\xi}{\kappa\alpha\eta\gamma\xi}, \tag{3.35}$$

and

$$I = \frac{(\alpha\eta)^2\delta\mu + \vartheta\delta(\alpha\eta - \gamma\xi)\gamma\xi - \kappa\alpha\eta\gamma\xi^2}{\kappa\alpha\eta^2\gamma\xi}. \tag{3.36}$$

Equation (3.34) is only feasible if $\beta\xi \leq \alpha\delta$ or equivelently if $\gamma\xi \leq \alpha\eta$. Now suppose this condition is satisfied, then a sufficient but not neccesary condition for I to be positive is

$$\delta\mu \geq \xi\kappa. \tag{3.37}$$

But $0 < \mu < 1$, and hence we have $\delta > \xi\kappa$. Now suppose all the above conditions are satisfied, then we have an interior steady state. Denote this state by

$$F_{5I} = (\frac{\mu(\alpha\eta)^2 + \vartheta(\alpha\eta - \gamma\xi)\gamma\xi}{\kappa\alpha\eta\gamma\xi}, \frac{(\alpha\eta)^2\delta\mu + \vartheta\delta(\alpha\eta - \gamma\xi)\gamma\xi - \kappa\alpha\eta\gamma\xi^2}{\kappa\alpha\eta^2\gamma\xi}, \frac{\gamma\xi}{\alpha\eta}).$$

3.3 Stability Analysis of Steady States

In this section, we shall calculate the Jacobian matrix at each steady state and study the local stability properties of each state. We shall therefore assume in this section

that the neccesary conditions for the existence of each steady state are satisfied. The Jacobian matrix J(A,I,E) for the system of Equations (3.5), (3.6) and (3.7) is given by

$$J(A,I,E) = \begin{bmatrix} \alpha E - 2\beta A + \gamma I & \gamma A & \alpha A \\ \delta I & -\xi - 2\eta I + \delta A & 0 \\ -\kappa E & 0 & -\kappa A + \vartheta(1-2E) \end{bmatrix}. \tag{3.38}$$

3.3.1 Stability of Axial Steady States

For the steady state located at the origin (i.e. $F_{1A} = (0,0,0)$), the Jacobian matrix (3.38) reduces to

$$J_{1A} = J(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\xi & 0 \\ 0 & 0 & \vartheta \end{bmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = 0$, $\lambda_2 = -\xi$ and $\lambda_3 = \vartheta$, which are real, λ_2 is always negative and λ_3 is always positive. Thus, this steady state is always of saddle type with respect to solutions initiating in the interior of the (I,E)-plane. It is locally unstable in the E-direction and locally stable in the I-direction.

For the steady state located on the E-axis (i.e $F_{2A} = (0,0,1)$), the Jacobian matrix (1.38) reduces to

$$J_{2A} = J(0,0,1) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & -\xi & 0 \\ -\kappa & 0 & -\vartheta \end{bmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = \alpha$, $\lambda_2 = -\xi$ and $\lambda_3 = -\vartheta$, which are real, λ_1 is always positive, λ_2 is always negative and λ_3 is always negative. Thus this steady state is of saddle type, which is locally unstable in the A-direction and locally stable

in both the I-direction and the E-direction. Thus the steady state corresponding to the maximum level of ecospheric assets is locally stable in the E-direction whereas the steady state corresponding to the minimum level of ecospheric assets is unstable in the E-direction.

3.3.2 Stability of the Interior Planar Steady States

We first consider the case of the planar states for which $\mu = 0$. For the steady states in the interior of the (A,I)-plane (i.e. F_{1P}), the Jacobian matrix (3.38) reduces to

$$J_{1P} = \frac{1}{\gamma\delta - \beta\eta} \begin{bmatrix} -\beta\gamma\xi & \gamma^2\xi & \alpha\gamma\xi \\ \delta\beta\xi & -\xi\beta\eta & 0 \\ 0 & 0 & \vartheta(\gamma\delta - \beta\eta) - \kappa\gamma\xi \end{bmatrix}.$$

The characteristic equation for the above matrix is given by

$$(\lambda + \frac{\kappa\gamma\xi}{\gamma\delta - \beta\eta} - \vartheta)(\lambda^2 - \tau_1\lambda + \Delta_1) = 0,$$

where $\tau_1 = \frac{-\beta\xi(\gamma+\xi)}{\gamma\delta-\beta\eta}$ and $\Delta_1 = -\frac{\beta\xi^2\gamma}{\gamma\delta-\beta\eta}$. The solutions are given by $\lambda_3 = -\frac{\kappa\gamma\xi}{\gamma\delta-\beta\eta} + \vartheta$ and

$$\lambda^2 - \tau_1\lambda + \Delta_1 = 0. \tag{3.39}$$

We can make the value of λ_3 positive, zero or negative depending on our choice of ϑ. The other two eigenvalues are the roots of equation (3.39) and are given by

$$\lambda_{1,2} = \frac{\tau_1 \pm \sqrt{\tau_1^2 - 4\Delta_1}}{2}.$$

Since $\Delta_1 < 0$, both λ_1 and λ_2 are real and of opposite signs. That is, one of λ_1 and λ_2 is positive and the other is negative. Thus F_{1P} is a saddle steady state with respect to solutions initiating in the interior of the (A,I)-plane. Hence it is a saddle in general.

For the steady state located in the interior of the (A,E)-plane (i.e. F_{2P}), the Jacobian matrix (3.38) reduces to

$$J_{2P} = \frac{1}{\kappa\alpha + \beta\vartheta}\begin{bmatrix} -\beta\alpha\vartheta & \gamma\alpha\vartheta & \alpha^2\vartheta \\ 0 & \delta\alpha\vartheta - \xi(\beta\vartheta + \kappa\alpha) & 0 \\ -\kappa\vartheta\alpha & 0 & -\vartheta^2\beta \end{bmatrix}.$$

The characteristic equation for the above matrix is given by

$$(\lambda - \frac{\delta\alpha\vartheta}{\kappa\alpha + \beta\vartheta} + \xi)(\lambda^2 - \tau_2\lambda + \Delta_2) = 0,$$

where $\tau_2 = \frac{-\beta\vartheta(\alpha+\vartheta)}{\kappa\alpha+\beta\vartheta}$ and $\Delta_2 = \frac{\alpha\vartheta^2\beta}{\kappa\alpha+\beta\vartheta}$. The solutions are given by $\lambda_2 = \frac{\delta\alpha\vartheta}{\kappa\alpha+\beta\vartheta} - \xi$ and

$$\lambda^2 - \tau_2\lambda + \Delta_2 = 0. \qquad (3.40)$$

We can make the value of λ_2 positive, negative or zero depending on our choice of ξ. The other two eigenvalues are given by the roots of (3.40) and are given by

$$\lambda_{1,3} = \frac{\tau_2 \pm \sqrt{\tau_2^2 - 4\Delta_2}}{2}.$$

Since $\tau_2 < 0$ and $\Delta_2 > 0$, both λ_1 and λ_3 may be real and negative or both may be complex with negative real parts. Thus F_{2P} is locally asymptotically stable with respect to solutions initiating in the interior of the (A,E)-plane.

We now show that there are no nontrivial closed path solutions to our system lying completely in the positive quadrant of the (A,E)-plane. In the positive quadrant of the (A,E)-plane, our system reduces to

$$\frac{dA}{dt} = \alpha EA - \beta A^2,$$

$$\frac{dE}{dt} = -\kappa EA + \vartheta(1 - E)E.$$

Now consider

$$D(A,E) = \frac{\partial(A^{-1}E^{-1}\frac{dA}{dt})}{\partial A} + \frac{\partial(A^{-1}E^{-1}\frac{dE}{dt})}{\partial E} = -\frac{\beta}{E} - \frac{\vartheta}{A} < 0,$$

for all (A,E) in the positive (A,E)-plane. Hence by Dulac's theorem (see [9],[21]), there are no closed path (periodic) solutions.

Now we consider the case of the interior planar state for which $\mu \neq 0$. This one occurs only in the (A,E)-plane and is given by F_{3P}. For this state, the Jacobian matrix (3.38) reduces to

$$J_{3P} = \begin{bmatrix} -\alpha E^* & \frac{\gamma\alpha E^*}{\beta} & \frac{\alpha^2 E^*}{\beta} \\ 0 & -\xi + \frac{\delta\alpha E^*}{\beta} & 0 \\ -\kappa E^* & 0 & -\frac{\kappa\alpha E^*}{\beta} + \vartheta(1-2E^*) \end{bmatrix},$$

where E^* is given by Equation (3.13). The characteristic equation for this matrix is given by

$$(\lambda - \frac{\delta\alpha E^*}{\beta} + \xi)(\lambda^2 - \tau_3\lambda + \Delta_3) = 0,$$

where $\tau_3 = \frac{(-\beta\alpha-\kappa\alpha-2\beta\vartheta)E^*+\beta\vartheta}{\beta}$ and $\Delta_3 = (\frac{2\kappa\alpha^2 E^*}{\beta} + 2\alpha\vartheta E^* - \alpha\vartheta)E^*$. The solutions are given by $\lambda_2 = \frac{\delta\alpha E^*}{\beta} - \xi$ and

$$\lambda^2 - \tau_3\lambda + \Delta_3 = 0. \tag{3.41}$$

We can make the value of λ_2 positive, zero or negative depending on our choice of ξ since E^* is does not depend on ξ. Also it is very easy to verify using Equation (3.13) that

$$E^* > \frac{\beta\vartheta}{\beta\vartheta + \kappa\alpha}. \tag{3.42}$$

Using the relation (3.42) as a minimal estimate of E^*, it easily follows that

$$\tau_3 = \frac{(-\beta\alpha - \kappa\alpha - 2\beta\vartheta)E^* + \beta\vartheta}{\beta} < 0$$

and

$$\Delta_3 = (\frac{2\kappa\alpha^2 E^*}{\beta} + 2\alpha\vartheta E^* - \alpha\vartheta)E^* > 0.$$

Hence the other roots of the characteristic equations are

$$\lambda_{1,3} = \frac{\tau_3 \pm \sqrt{\tau_3^2 - 4\Delta_3}}{2},$$

which are both either negative real numbers or complex numbers with negative real parts. Thus F_{3P} is locally asymptotically stable with respect to solutions initiating from the interior of the (A,E)-plane.

It also follows just as in the case for $\mu = 0$ that there are no nontrivial closed path solutions to our system lying completely in the positive quadrant of the (A,E)-plane.

3.3.3 Stability of the Interior Steady States

Here we study the local stability of each of the interior steady states considered in Section 3.2.3. We therefore assume throughout this section that the neccesary conditions which quarantee that the various interior steady states exist described in Section 3.2.3 are satisfied. In order to make our analysis easier, we will make use of the following facts.

Lemma 3.1:

The characteristic equation of a matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

is given by

$$\lambda^3 + \tau_2\lambda^2 + \tau_1\lambda + \tau_0 = 0, \tag{3.43}$$

where

$$\tau_2 = -b_{11} - b_{22} - b_{33},$$

$$\tau_1 = b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31},$$

$$\tau_0 = b_{12}b_{21}b_{33} + b_{13}b_{31}b_{22} - b_{11}b_{22}b_{33} - b_{12}b_{31}b_{23} - b_{13}b_{21}b_{32}.$$

Lemma 3.2: (The Routh-Hurwitz Criterion)

A neccesary and sufficient condition for all the eigenvalues of the matrix B above (i.e. for all the roots of equation (3.43)) to have negative real parts is that

$$(i) \quad \tau_2 > 0, \qquad (ii) \quad \tau_0 > 0, \quad \text{and} \qquad (iii) \quad \tau_2\tau_1 - \tau_0 > 0.$$

Proof : (See ([1],[7],[26]))

Now we consider F_{1I}. This steady state exists if $\gamma = \mu = 0$ and $\delta\alpha\vartheta - \alpha\kappa\xi - \beta\vartheta\xi > 0$. For this steady state, the Jacobian matrix (3.38) reduces to

$$J_{1I} = \frac{1}{\alpha\kappa + \beta\vartheta} \begin{bmatrix} -\alpha\beta\vartheta & 0 & \alpha^2\vartheta \\ \frac{\Delta\delta}{\eta} & -\Delta & 0 \\ -\kappa\beta\vartheta & 0 & -\beta\vartheta^2 \end{bmatrix},$$

where $\Delta = \delta\alpha\vartheta - \alpha\kappa\xi - \beta\vartheta\xi$. The characteristic equation of J_{1I} by Lemma 3.1 is given by

$$\lambda^3 + \tau_2\lambda^2 + \tau_1\lambda + \tau_0 = 0,$$

where

$$\tau_2 = \frac{\alpha\beta\vartheta + \Delta + \beta\vartheta^2}{\alpha\kappa + \beta\vartheta},$$

$$\tau_1 = \frac{\Delta(\alpha\beta\vartheta + \beta\vartheta^2) + \alpha\beta^2\vartheta^3 + \kappa\alpha^2\beta\vartheta^2}{(\alpha\kappa + \beta\vartheta)^2},$$

$$\tau_0 = \frac{\alpha\beta\vartheta^2\Delta}{(\alpha\kappa + \beta\vartheta)^2}.$$

Clearly $\tau_2 > 0, \quad \tau_2 > 0 \quad and \quad \tau_3 > 0$. Now consider

$$\tau_2\tau_1 - \tau_0 = \frac{(\alpha\beta\vartheta + \beta\vartheta^2)(\Delta + \alpha\beta^2\vartheta^3 + \kappa\alpha^2\beta\vartheta^2) + \alpha\beta\vartheta + \beta\vartheta + \Delta}{(\alpha\kappa + \beta\vartheta)^3} > 0.$$

Hence by Lemma 3.2, the steady state F_{1I} is locally asymptotically stable, since all the eigenvalues of the corresponding Jacobian matrix have negative real parts.

Now we consider the case of the interior steady state where $\gamma < 0$ and $\mu = 0$ (i.e. F_{2I}). We recall that this steady state exists under the following conditions

$$(i) \quad \gamma < 0 \quad \text{and} \quad (ii) \quad \delta\alpha\vartheta - \alpha\kappa\xi - \beta\vartheta\xi < 0.$$

It should be noted that condition (ii) also implies that $\beta\eta\vartheta + \kappa\gamma\xi > \gamma\delta\vartheta$. For this steady state, the Jacobian matrix (3.38) reduces to

$$J_{2I} = \frac{1}{\Delta_4}\begin{bmatrix} -\beta\vartheta\Delta_1 & \gamma\vartheta\Delta_1 & \alpha\vartheta\Delta_1 \\ \alpha\vartheta\delta^2 & -\eta\Delta_2 & 0 \\ -\kappa\Delta_3 & 0 & -\vartheta\Delta_3 \end{bmatrix},$$

where

$$\Delta_1 = \alpha\eta - \gamma\xi, \quad \Delta_2 = \delta\alpha\vartheta - \alpha\kappa\xi - \beta\vartheta\xi$$

$$\Delta_3 = \beta\eta\vartheta + \kappa\gamma\xi - \gamma\delta\vartheta, \quad \text{and} \quad \Delta_4 = \alpha\eta\kappa + \vartheta\beta\eta - \vartheta\gamma\delta.$$

Using Lemma 3.1, the characteristic equation of J_{2I} is given by

$$\lambda^3 + \tau_2\lambda^2 + \tau_1\lambda + \tau_0 = 0,$$

where

$$\tau_2 = \frac{\beta\vartheta\Delta_1 + \eta\Delta_2 + \vartheta\Delta_3}{\Delta_4}, \tag{3.44}$$

$$\tau_1 = \frac{\beta\eta\vartheta\Delta_1\Delta_2 + \beta\vartheta^2\Delta_1\Delta_3 + \eta\vartheta\Delta_2\Delta_3 - \alpha\vartheta^2\delta^2\gamma\Delta_1 + \alpha\kappa\vartheta\Delta_1\Delta_3}{\Delta_4^2}, \tag{3.45}$$

$$\tau_0 = \frac{\Delta_1\Delta_2\Delta_3(\alpha\kappa\eta\vartheta + \beta\eta\vartheta^2) - \alpha\vartheta^3\delta^2\gamma\Delta_1\Delta_3}{\Delta_4^3}. \tag{3.46}$$

Clearly

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \Delta_3 > 0 \quad \text{and} \quad \Delta_4 > 0.$$

Hence $\tau_2 > 0$, $\tau_1 > 0$ and $\tau_0 > 0$. Now consider

$$\begin{aligned}\tau_2\tau_1 - \tau_0 = {} & (1/\Delta_4^3)[(\beta\vartheta\Delta_1 + \eta\Delta_2 + \vartheta\Delta_3)(\beta\vartheta\eta\Delta_1\Delta_2 + \eta\vartheta\Delta_2\Delta_3) \\ & + (\beta\vartheta\Delta_1 + \vartheta\Delta_3)(\beta\vartheta^2 + \alpha\kappa\vartheta)\Delta_1\Delta_3 - \alpha\vartheta^2\delta^2\gamma\Delta_1(\beta\vartheta\Delta_1 + \eta\Delta_2)], \\ & > 0.\end{aligned} \tag{3.47}$$

Hence by Lemma 3.2, F_{2I} is locally asymptotically stable.

Now we consider the case of the interior steady state where $\gamma > 0$ and $\mu = 0$. We recall that in this case, we have an interior steady state given by F_{2I} (i.e. same as the case with $\gamma < 0$ and $\mu = 0$) provided either one of the following conditions is satisfied.

$$(i) \qquad \frac{\alpha\vartheta\delta}{\xi} < \alpha\kappa + \beta\vartheta < \frac{\vartheta\gamma\delta}{\eta}.$$

$$(ii) \qquad \frac{\alpha\vartheta\delta}{\xi} > \alpha\kappa + \beta\vartheta > \frac{\vartheta\gamma\delta}{\eta}.$$

Thus for this steady state the Jacobian matrix (3.38) reduces to J_{2I} with characteristic equation given by Equation (3.43), where τ_2, τ_1, and τ_0 are given respectively by Equations (3.44), (3.45) and (3.46) and Δ_1, Δ_2, Δ_3 and Δ_4 are as defined in the previous paragraph.

Now suppose Condition (i) is satisfied (i.e. $\frac{\alpha\vartheta\delta}{\xi} < \alpha\kappa + \beta\vartheta < \frac{\vartheta\gamma\delta}{\eta}$ and $\gamma > 0$), then we have

$$\Delta_1 = \alpha\eta - \gamma\xi < 0,$$

$$\Delta_2 = \delta\alpha\vartheta - \alpha\kappa\xi - \beta\vartheta\xi < 0,$$

$$\Delta_3 = \beta\eta\vartheta + \kappa\gamma\xi - \gamma\delta\vartheta < 0$$

and

$$\Delta_4 = \alpha\eta\kappa + \vartheta\beta\eta - \vartheta\gamma\delta < 0.$$

Substituting the above into Equations (3.44), (3.45), (3.46) and (3.47), we obtain $\tau_2 > 0$, $\tau_1 > 0$, $\tau_0 > 0$ and $\tau_2\tau_1 - \tau_0 > 0$. Hence by Lemma 3.2, F_{2I} is locally asymptotically stable if Condition (i) is satisfied.

If on the other hand, Condition (ii) is satisfied (i.e. $\frac{\alpha\vartheta\delta}{\xi} > \alpha\kappa + \beta\vartheta > \frac{\vartheta\gamma\delta}{\eta}$ and $\gamma > 0$), then we have

$$\Delta_1 = \alpha\eta - \gamma\xi > 0,$$

$$\Delta_2 = \delta\alpha\vartheta - \alpha\kappa\xi - \beta\vartheta\xi > 0,$$

$$\Delta_3 = \beta\eta\vartheta + \kappa\gamma\xi - \gamma\delta\vartheta > 0$$

and

$$\Delta_4 = \alpha\eta\kappa + \vartheta\beta\eta - \vartheta\gamma\delta > 0.$$

Substituting the above into Equations (1.44), we have $\tau_2 > 0$.

Now we consider the case of the interior steady state where $\gamma = 0$ and $\mu > 0$. In this case the steady state is given by F_{3I} provided $\frac{\beta\xi}{\alpha\delta} < E_{1I}^* \leq 1$. For this steady state, the Jacobian matrix (3.38) reduces to

$$J_{3I} = \begin{bmatrix} -\alpha E_{1I}^* & 0 & \frac{\alpha^2 E_{1I}^*}{\beta} \\ \frac{-\delta\beta\xi + \delta^2\alpha E_{1I}^*}{\eta\beta} & \frac{\xi\beta - \delta\alpha E_{1I}^*}{\beta} & 0 \\ -\kappa E_{1I}^* & 0 & \frac{(-\kappa\alpha - 2\vartheta\beta)E_{1I}^* + \vartheta\beta}{\beta} \end{bmatrix},$$

where E_{1I}^* is given by Equation (3.27). The characteristic equation of this matrix is given by

$$(\lambda + \frac{\delta\alpha E_{1I}^* - \xi\beta}{\beta})(\lambda^2 - \tau\lambda + \Delta) = 0,$$

where

$$\tau = \frac{(-\alpha\beta - \alpha\kappa - 2\vartheta\beta)E_{1I}^* + \vartheta\beta}{\beta}$$

and

$$\Delta = \frac{[(2\kappa\alpha^2 + 2\vartheta\alpha\beta)E_{1I}^* - \vartheta\alpha\beta]E_{1I}^*}{\beta}.$$

The solutions are given by

$$\lambda_2 = \frac{-\delta\alpha E_{1I}^* + \xi\beta}{\beta}$$

and

$$\lambda^2 - \tau\lambda + \Delta = 0.$$

Clearly, $\lambda_2 < 0$ since $\frac{\beta\xi}{\alpha\delta} < E_{1I}^* \leq 1$. The other two eigenvalues are given by

$$\lambda_{1,3} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}.$$

But it is obvious from equation (3.27) that $E_{1I}^* > \frac{\beta\vartheta}{\beta\vartheta + \alpha\kappa}$, and hence

$$\begin{aligned}
\tau &= \frac{(-\alpha\beta - \alpha\kappa - 2\vartheta\beta)E_{1I}^* + \vartheta\beta}{\beta} \\
&< \frac{(-\alpha\beta - \alpha\kappa - 2\vartheta\beta)\frac{\beta\vartheta}{\beta\vartheta + \alpha\kappa} + \vartheta\beta}{\beta} \\
&= -\frac{\vartheta\beta(\alpha + \vartheta)}{\beta\vartheta + \kappa\alpha} < 0.
\end{aligned}$$

Also

$$\begin{aligned}
\Delta &= \frac{[(2\kappa\alpha^2 + 2\vartheta\alpha\beta)E_{1I}^* - \vartheta\alpha\beta]E_{1I}^*}{\beta} \\
&> \frac{[(2\kappa\alpha^2 + 2\vartheta\alpha\beta)\frac{\beta\vartheta}{\beta\vartheta + \alpha\kappa} - \vartheta\alpha\beta]E_{1I}^*}{\beta} \\
&= \alpha\vartheta E_{1I}^* > 0.
\end{aligned}$$

Thus both λ_1 and λ_3 may be real and negative or both may be complex with negative real parts (because $\tau < 0$ and $\Delta > 0$). Hence F_{3I} is locally asymptotically stable if it exists.

Now we consider the case of the interior steady state where $\gamma < 0$ and $\mu > 0$. In this case we have an equilibrium given by

$$F_{4I}^{+} = \left(\frac{\alpha\eta E_{2I}^{+} - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\delta\alpha E_{2I}^{+} - \beta\xi}{\beta\eta - \gamma\delta}, E_{2I}^{+}\right),$$

where E_{2I}^{+} is given by Equation (3.31), provided $\frac{\beta\xi}{\delta\alpha} < E_{2I}^{+} \leq 1$. At this point the Jacobian matrix (3.38) reduces to

$$J_{4I}^{+} = \begin{bmatrix} -\beta\Delta_1^{+} & \gamma\Delta_1^{+} & \alpha\Delta_1^{+} \\ \delta\Delta_2^{+} & -\eta\Delta_2^{+} & 0 \\ -\kappa E_{2I}^{+} & 0 & -\Delta_3^{+} \end{bmatrix},$$

where

$$\Delta_1^{+} = \frac{\alpha\eta E_{2I}^{+} - \gamma\xi}{\beta\eta - \gamma\delta} > 0, \qquad \Delta_2^{+} = \frac{\delta\alpha E_{2I}^{+} - \beta\xi}{\beta\eta - \gamma\delta} > 0,$$

and

$$\Delta_3^{+} = \frac{(\kappa\alpha\eta + 2\vartheta(\beta\eta - \gamma\delta))E_{2I}^{+} - (\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta))}{\beta\eta - \gamma\delta}.$$

We now show that $\Delta_3^{+} > 0$. If $\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta) \leq 0$, then the result follows from the expression for Δ_3^{+}. If on the other hand $\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta) > 0$, then from Equation (3.31) we have

$$E_{2I}^{+} > \frac{\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta)}{\kappa\alpha\eta + \vartheta(\beta\eta - \gamma\delta)} > \frac{\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta)}{\kappa\alpha\eta + 2\vartheta(\beta\eta - \gamma\delta)},$$

and hence $\Delta_3^{+} > 0$. The characteristic equation for the above matrix, J_{4I}^{+} (using Lemma 3.1) is given by

$$\lambda^3 + \tau_2^{+}\lambda^2 + \tau_1^{+}\lambda + \tau_0^{+} = 0,$$

where

$$\tau_2^{+} = \beta\Delta_1^{+} + \eta\Delta_2^{+} + \Delta_3^{+} > 0,$$

$$\tau_1^{+} = (\beta\eta - \gamma\delta)\Delta_1^{+}\Delta_2^{+} + \beta\Delta_1^{+}\Delta_3^{+} + \eta\Delta_2^{+}\Delta_3^{+} + \alpha\kappa E_{2I}^{+}\Delta_1^{+} > 0,$$

$$\tau_0^+ = (\beta\eta - \gamma\delta)\Delta_1^+\Delta_2^+\Delta_3^+ + \alpha\eta\kappa E_{2I}^+\Delta_1^+\Delta_2^+ > 0.$$

Also

$$\tau_2^+\tau_1^+ - \tau_0^+ = (\beta\Delta_1^+ + \eta\Delta_2^+ + \Delta_3^+)(\beta\Delta_1^+\Delta_3^+\eta\Delta_2^+\Delta_3^+) + (\beta\eta - \gamma\delta)\Delta_1^+\Delta_2^+(\beta\Delta_1^+ + \eta\Delta_2^+)$$
$$+ \alpha\kappa E_{2I}^+\Delta_1^+(\beta\Delta_1^+ + \Delta_3^+) > 0,$$

since $\Delta_3^+ > 0$, $\Delta_2^+ > 0$ and $\Delta_1^+ > 0$. Hence F_{4I}^+ is locally asymptotically stable (by Lemma 3.2).

We now consider the case of the interior steady state where $\gamma > 0$, $\mu > 0$. Here we have to consider three different cases: (i) $\gamma > 0$, $\mu > 0$ and $\beta\eta - \gamma\delta > 0$, (ii) $\gamma > 0$, $\mu > 0$ and $(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha < 0$, and (iii) $\gamma > 0$, $\mu > 0$, $(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha > 0$, $(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi > 0$, $\beta\eta - \gamma\delta < 0$ and

$$[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha]^2 + 4\mu(\beta\eta - \gamma\delta)[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha] \geq 0.$$

Case I: $\gamma > 0$, $\mu > 0$ and $\beta\eta - \gamma\delta > 0$.

In this case, we have an interior steady state given by

$$F_{4I}^+ = (\frac{\alpha\eta E_{2I}^+ - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\delta\alpha E_{2I}^+ - \beta\xi}{\beta\eta - \gamma\delta}, E_{2I}^+),$$

where E_{2I}^+ is given by Equation (3.31), provided $\frac{\beta\xi}{\delta\alpha} < E_{2I}^+ \leq 1$. For this steady state the Jacobian matrix (3.38) reduces to J_{4I}^+ (i.e. same as the case of $\gamma < 0$, $\mu > 0$). In this case, we observe that $(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi > 0$, and hence $\Delta_3^+ > 0$ (i.e. similar to the case of $\gamma < 0$, $\mu > 0$). Hence it follow that all the eigenvalues of the matrix J_{4I}^+ corresponding to this steady state will all have negative real parts. Thus, by Lemma 3.2, F_{4I}^+ is locally asymptotically stable.

Case II: $\gamma > 0$, $\mu > 0$ and $(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha < 0$.

In this case, we have a steady state given by

$$F_{4I}^- = (\frac{\alpha\eta E_{2I}^- - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\delta\alpha E_{2I}^- - \beta\xi}{\beta\eta - \gamma\delta}, E_{2I}^-),$$

where E_{2I}^- is given by Equation (3.31), provided $\frac{\beta\xi}{\delta\alpha} > E_{2I}^- > 0$. For this steady state the Jacobian matrix (3.38) reduces to

$$J_{4I}^- = \begin{bmatrix} -\beta\Delta_1^- & \gamma\Delta_1^- & \alpha\Delta_1^- \\ \delta\Delta_2^- & -\eta\Delta_2^- & 0 \\ -\kappa E_{2I}^- & 0 & -\Delta_3^- \end{bmatrix},$$

where

$$\Delta_1^- = \frac{\alpha\eta E_{2I}^- - \gamma\xi}{\beta\eta - \gamma\delta} > 0,$$

$$\Delta_2^- = \frac{\delta\alpha E_{2I}^- - \beta\xi}{\beta\eta - \gamma\delta} > 0,$$

and

$$\Delta_3^- = \frac{(\kappa\alpha\eta + 2\vartheta(\beta\eta - \gamma\delta))E_{2I}^- - (\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta))}{\beta\eta - \gamma\delta}.$$

If $\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta) \geq 0$, then $\Delta_3^- > 0$. If on the other hand $\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta) < 0$, then from Equation (3.31) we have $E_{2I}^- > \frac{\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta)}{\kappa\eta\alpha + \vartheta(\beta\eta - \gamma\delta)}$. Therefore

$$\begin{aligned} \Delta_3^- &= \frac{(\kappa\alpha\eta + 2\vartheta(\beta\eta - \gamma\delta))E_{2I}^- - (\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta))}{\beta\eta - \gamma\delta} \\ &> \frac{(\kappa\alpha\eta + 2\vartheta(\beta\eta - \gamma\delta))(\frac{\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta)}{\kappa\eta\alpha + \vartheta(\beta\eta - \gamma\delta)}) - (\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta))}{\beta\eta - \gamma\delta} \\ &= \vartheta\frac{\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta)}{\kappa\eta\alpha + \vartheta(\beta\eta - \gamma\delta)} > 0. \end{aligned}$$

Thus we have $\Delta_1^- > 0$, $\Delta_2^- > 0$ and $\Delta_3^- > 0$. The characteristic equation for this matrix J_{4I}^-, is given by (using Lemma 3.1)

$$\lambda^3 + \tau_2^-\lambda^2 + \tau_1^-\lambda + \tau_0^- = 0,$$

where

$$\tau_2^- = \beta\Delta_1^- + \eta\Delta_2^- + \Delta_3^- > 0,$$

$$\tau_1^- = (\beta\eta - \gamma\delta)\Delta_1^-\Delta_2^- + \beta\Delta_1^-\Delta_3^- + \eta\Delta_2^-\Delta_3^- + \alpha\kappa E_{2I}^-\Delta_1^-,$$

$$\tau_0^- = (\beta\eta - \gamma\delta)\Delta_1^-\Delta_2^-\Delta_3^- + \alpha\eta\kappa E_{2I}^-\Delta_1^-\Delta_2^-.$$

If $(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi \geq 0$, then

$$\begin{aligned}
\tau_0^- &= (\beta\eta - \gamma\delta)\Delta_1^-\Delta_2^-\Delta_3^- + \alpha\eta\kappa E_{2I}^-\Delta_1^-\Delta_2^- \\
&= \Delta_1^-\Delta_2^-[(\beta\eta - \gamma\delta)\Delta_3^- + \alpha\eta\kappa E_{2I}^-] \\
&= \Delta_1^-\Delta_2^-[2((\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha)E_{2I}^- - ((\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi)] < 0.
\end{aligned}$$

However, if $(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi < 0$, then $E_{2I}^- > \frac{\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta)}{\kappa\eta\alpha + \vartheta(\beta\eta - \gamma\delta)}$. Hence

$$\begin{aligned}
\tau_0^- &= (\beta\eta - \gamma\delta)\Delta_1^-\Delta_2^-\Delta_3^- + \alpha\eta\kappa E_{2I}^-\Delta_1^-\Delta_2^- \\
&= \Delta_1^-\Delta_2^-[(\beta\eta - \gamma\delta)\Delta_3^- + \alpha\eta\kappa E_{2I}^-] \\
&= \Delta_1^-\Delta_2^-[2((\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha)E_{2I}^- - ((\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi)] \\
&< \Delta_1^-\Delta_2^-[2((\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi) - ((\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi)] \\
&= \Delta_1^-\Delta_2^-[(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi] < 0.
\end{aligned}$$

Thus $\tau_0^- < 0$ for any choice of parameters in this case. We see (by Lemma 3.2) that F_{4I}^- is locally unstable.

Case III: $\gamma > 0$, $\mu > 0$, $(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha > 0$, $(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi > 0$, $\beta\eta - \gamma\delta < 0$ and

$$[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha]^2 + 4\mu(\beta\eta - \gamma\delta)[(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha] \geq 0.$$

For this case, we have two steady states given by

$$F_{4I}^{\pm} = \left(\frac{\alpha\eta E_{2I}^{\pm} - \gamma\xi}{\beta\eta - \gamma\delta}, \frac{\delta\alpha E_{2I}^{\pm} - \beta\xi}{\beta\eta - \gamma\delta}, E_{2I}^{\pm}\right)$$

provided $0 < E_{2I}^{\pm} < \frac{\beta\xi}{\alpha\delta}$. For these states, the Jacobian matrix (3.38) reduces to

$$J_{4I}^{\pm} = \begin{bmatrix} -\beta\Delta_1^{\pm} & \gamma\Delta_1^{\pm} & \alpha\Delta_1^{\pm} \\ \delta\Delta_2^{\pm} & -\eta\Delta_2^{\pm} & 0 \\ -\kappa E_{2I}^{\pm} & 0 & -\Delta_3^{\pm} \end{bmatrix},$$

where

$$\Delta_1^{\pm} = \frac{\alpha\eta E_{2I}^{\pm} - \gamma\xi}{\beta\eta - \gamma\delta} > 0, \qquad \Delta_2^{\pm} = \frac{\delta\alpha E_{2I}^{\pm} - \beta\xi}{\beta\eta - \gamma\delta} > 0,$$

and

$$\Delta_3^{\pm} = \frac{(\kappa\alpha\eta + 2\vartheta(\beta\eta - \gamma\delta))E_{2I}^{\pm} - (\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta))}{\beta\eta - \gamma\delta},$$

where $E_{2I}^{\pm}$ is given by Equation (3.31). The characteristic equations of $J_{4I}^{\pm}$ are given by (using Lemma 3.1)

$$\lambda^3 + \tau_2^{\pm}\lambda^2 + \tau_1^{\pm}\lambda + \tau_0^{\pm} = 0,$$

where

$$\tau_2^{\pm} = \beta\Delta_1^{\pm} + \eta\Delta_2^{\pm} + \Delta_3^{\pm},$$

$$\tau_1^{\pm} = (\beta\eta - \gamma\delta)\Delta_1^{\pm}\Delta_2^{\pm} + \beta\Delta_1^{\pm}\Delta_3^{\pm} + \eta\Delta_2^{\pm}\Delta_3^{\pm} + \alpha\kappa E_{2I}^{\pm}\Delta_1^{\pm},$$

$$\tau_0^{\pm} = (\beta\eta - \gamma\delta)\Delta_1^{\pm}\Delta_2^{\pm}\Delta_3^{\pm} + \alpha\eta\kappa E_{2I}^{\pm}\Delta_1^{\pm}\Delta_2^{\pm}.$$

From Equation (3.31) we have

$$E_{2I}^{+} < \frac{(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi}{(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha}$$

and

$$E_{2I}^{-} < \frac{(\beta\eta - \gamma\delta)\vartheta + \kappa\gamma\xi}{2(\beta\eta - \gamma\delta)\vartheta + 2\kappa\eta\alpha}.$$

Therefore,

$$\begin{aligned}
\Delta_3^{+} &= \frac{(\kappa\alpha\eta + 2\vartheta(\beta\eta - \gamma\delta))E_{2I}^{+} - (\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta))}{\beta\eta - \gamma\delta} \\
&= \frac{(\kappa\alpha\eta + \vartheta(\beta\eta - \gamma\delta))E_{2I}^{+} - (\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta))}{\beta\eta - \gamma\delta} + \vartheta E_{2I}^{+} \\
&> \frac{(\kappa\alpha\eta + \vartheta(\beta\eta - \gamma\delta))(\frac{(\beta\eta-\gamma\delta)\vartheta + \kappa\gamma\xi}{(\beta\eta-\gamma\delta)\vartheta + \kappa\eta\alpha}) - (\kappa\gamma\xi + \vartheta(\beta\eta - \gamma\delta))}{\beta\eta - \gamma\delta} + \vartheta E_{2I}^{+} \\
&= \vartheta E_{2I}^{+} > 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Delta_3^- &= \frac{(\kappa\alpha\eta + 2\vartheta(\beta\eta-\gamma\delta))E_{2I}^- - (\kappa\gamma\xi + \vartheta(\beta\eta-\gamma\delta))}{\beta\eta-\gamma\delta} \\
&= \frac{(\kappa\alpha\eta + \vartheta(\beta\eta-\gamma\delta))E_{2I}^- - (\kappa\gamma\xi + \vartheta(\beta\eta-\gamma\delta))}{\beta\eta-\gamma\delta} + \vartheta E_{2I}^- \\
&> \frac{(\kappa\alpha\eta + \vartheta(\beta\eta-\gamma\delta))(\frac{(\beta\eta-\gamma\delta)\vartheta+\kappa\gamma\xi}{2(\beta\eta-\gamma\delta)\vartheta+2\kappa\eta\alpha}) - (\kappa\gamma\xi + \vartheta(\beta\eta-\gamma\delta))}{\beta\eta-\gamma\delta} + \vartheta E_{2I}^- \\
&= -\frac{(\kappa\gamma\xi + \vartheta(\beta\eta-\gamma\delta))}{2(\beta\eta-\gamma\delta)} + \vartheta E_{2I}^- > 0.
\end{aligned}$$

Hence,

$$\tau_2^\pm = \beta\Delta_1^\pm + \eta\Delta_2^\pm + \Delta_3^\pm > 0.$$

Also

$$\begin{aligned}
\tau_0^\pm &= (\beta\eta-\gamma\delta)\Delta_1^\pm\Delta_2^\pm\Delta_3^\pm + \alpha\eta\kappa E_{2I}^\pm\Delta_1^\pm\Delta_2^\pm \\
&= \Delta_1^\pm\Delta_2^\pm[2((\beta\eta-\gamma\delta)\vartheta + \kappa\eta\alpha)E_{2I}^\pm - ((\beta\eta-\gamma\delta)\vartheta + \kappa\eta\alpha)].
\end{aligned}$$

But we have from Equation (3.31) that

$$E_{2I}^+ > \frac{(\beta\eta-\gamma\delta)\vartheta + \kappa\gamma\xi}{2(\beta\eta-\gamma\delta)\vartheta + 2\kappa\eta\alpha}$$

and

$$E_{2I}^- < \frac{(\beta\eta-\gamma\delta)\vartheta + \kappa\gamma\xi}{2(\beta\eta-\gamma\delta)\vartheta + 2\kappa\eta\alpha}.$$

Hence

$$\begin{aligned}
\tau_0^+ &> \Delta_1^+\Delta_2^+[2((\beta\eta-\gamma\delta)\vartheta + \kappa\eta\alpha)(\frac{(\beta\eta-\gamma\delta)\vartheta + \kappa\gamma\xi}{2(\beta\eta-\gamma\delta)\vartheta + 2\kappa\eta\alpha}) - ((\beta\eta-\gamma\delta)\vartheta + \kappa\eta\alpha)] \\
&= 0.
\end{aligned}$$

It can be shown similarly that $\tau_0^- < 0$. Thus F_{4I}^- is locally unstable since Lemma 3.2 has been violated. We proceed to determine the local stability of F_{4I}^+ below. We are not yet able to show that

$$\tau_2^+\tau_1^+ - \tau_0^+ > 0,$$

but we have observed from the numerical solutions that anytime F^+_{4I} exists, it appears to be locally asymptotically stable.

3.4 Numerical Examples

In this section, we use XPP software to study the phase portrait of our system by choosing values for the model parameters. These values have been arbitrarily chosen subject only to the assumptions of our model, and do not represent any actual agriculture- industry-ecospheric system. However, for consistency, the values chosen in this section will be similar to those in the previous chapter. We choose

$$\alpha = 8.0, \quad \beta = 2.0, \quad \eta = 0.1, \quad \delta = 0.75, \quad \kappa = 0.5, \quad \vartheta = 2.0.$$

The other parameter values are given for each figure below.

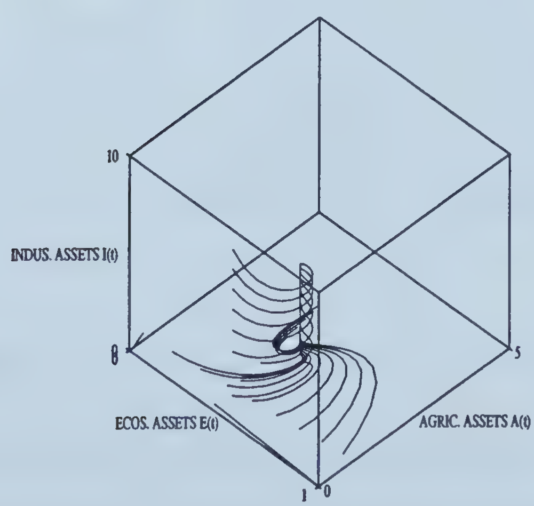


Figure 3.1: Phase portrait for system when $\gamma = 0.0$ and $\xi = 1.0$ and $\mu = 0$.

In Figure 3.1, we chose $\mu = 0$, $\gamma = 0$ and $\xi = 1$. In this case we have two axial steady states F_{1A} and F_{2A}, one planar state F_{2P} lying in the interior of the (A-E)-plane and one interior steady state F_{1I} since $\delta\alpha\vartheta > \xi(\alpha\kappa + \beta\vartheta)$. The figure illustrates the fact that F_{1A}, F_{2A} and F_{2P} are all locally unstable and F_{1I} is locally asymptotically stable. This figure suggests that F_{1I} is globally asymptotically stable. This shows that if there is no net loss or gain of assets by agriculture from industry as a result of their interaction, and the constant rate of depreciation of industrial assets is small so that $\delta\alpha\vartheta > \xi(\alpha\kappa + \beta\vartheta)$, then none of the assets will go extinct if the correction rate of ecospheric assets is zero.

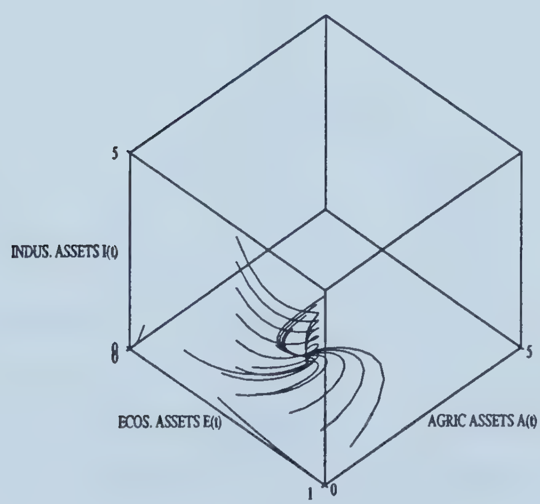


Figure 3.2: Phase portrait for system when $\gamma = 0.0$ and $\xi = 2.0$ and $\mu = 0$.

In Figure 3.2, we chose $\mu = 0$, $\gamma = 0$ and $\xi = 2$. In this case we have two axial steady states F_{1A} and F_{2A}, one planar state F_{2P} lying in the interior of the (A-E)-plane and no interior steady state since $\delta\alpha\vartheta < \xi(\alpha\kappa + \beta\vartheta)$. The figure illustrates the fact that

F_{1A}, F_{2A} are all locally unstable and F_{2P} is locally asymptotically stable. This figure suggests that F_{2p} is globally asymptotically stable. This shows that if there is no net loss or gain of assets by agriculture from industry as a result of their interaction, and the constant rate of depreciation of industrial assets is large so that $\delta\alpha\vartheta < \xi(\alpha\kappa+\beta\vartheta)$, then industrial assets will go extinct if the correction rate of ecospheric assets is zero. Figures 3.1 and 3.2 shows that if the interaction between agricultural, industrial and ecospheric assets is such that there is no net gain or loss of assets by agriculture from industry and there is no correction to the loss of ecospheric assets to agriculture, then the system will persist or co-exist if only the constant depreciation rate of industrial assets is small so that $\delta\alpha\vartheta > \xi(\alpha\kappa + \beta\vartheta)$.

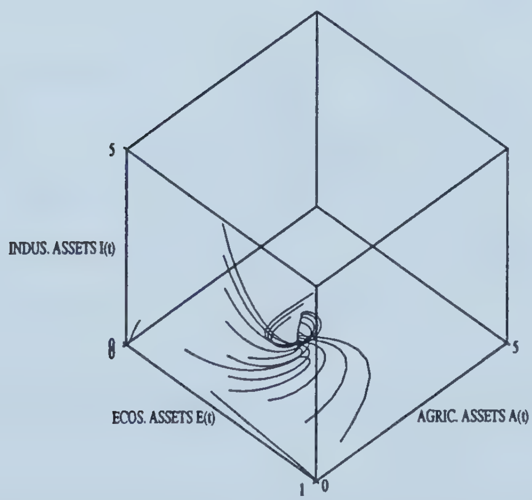


Figure 3.3: Phase portrait for system when $\gamma = -1.0$ and $\xi = 1.0$ and $\mu = 0$.

In Figure 3.3, we chose $\mu = 0$, $\gamma = -1.0$ and $\xi = 1$. For this case we have two axial steady states F_{1A} and F_{2A}, one planar state F_{2P} lying in the interior of the

(A-E)-plane and one interior steady state F_{2I} since $\delta\alpha\vartheta > \xi(\alpha\kappa + \beta\vartheta)$. The figure illustrates the fact that F_{1A}, F_{2A} and F_{2P} are all locally unstable and F_{2I} is locally asymptotically stable. This figure suggests that F_{2I} is globally asymptotically stable. This shows that if there is a net loss of assets by agriculture to industry as a result of their interaction and the constant rate of depreciation of industrial assets is small such that $\delta\alpha\vartheta > \xi(\alpha\kappa + \beta\vartheta)$, then none of the assets will go extinct if the correction rate of ecospheric assets is zero.

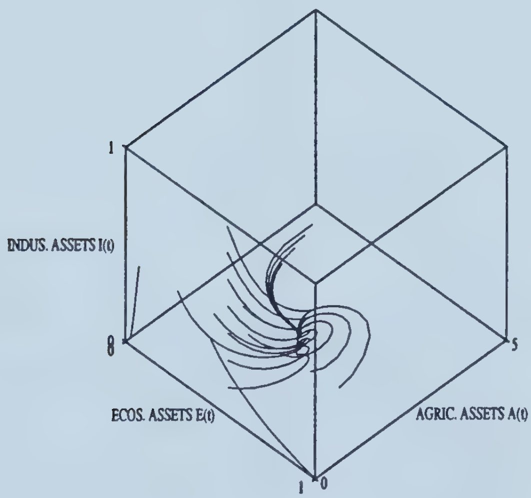


Figure 3.4: Phase portrait for system when $\gamma = -1.0$ and $\xi = 2.0$ and $\mu = 0$.

In Figure 3.4, we chose $\mu = 0$, $\gamma = -1.0$ and $\xi = 2$. In this case we have two axial steady states F_{1A} and F_{2A}, one planar state F_{2P} lying in the interior of the (A-E)-plane and no interior steady state since $\delta\alpha\vartheta < \xi(\alpha\kappa + \beta\vartheta)$. The figure illustrates the fact that F_{1A}, F_{2A} are all locally unstable and F_{2P} is locally asymptotically stable. This figure suggests that F_{2P} is globally asymptotically stable. This shows that if there is

a net loss of assets by agriculture to industry as a result of their interaction and the constant rate of depreciation of industrial assets is large so that $\delta\alpha\vartheta < \xi(\alpha\kappa + \beta\vartheta)$, then industrial assets will go extinct if there is no correction to the loss of ecospheric assets to agriculture.

Figures 3.3 and 3.4 show that if the interaction between agricultural, industrial and ecospheric assets is such that there is a net loss of assets by agriculture to industry and there is no correction to the loss of ecospheric assets to agriculture, then the system will persist or co-exist if the constant depreciation rate of industrial assets is small so that $\delta\alpha\vartheta > \xi(\alpha\kappa + \beta\vartheta)$.

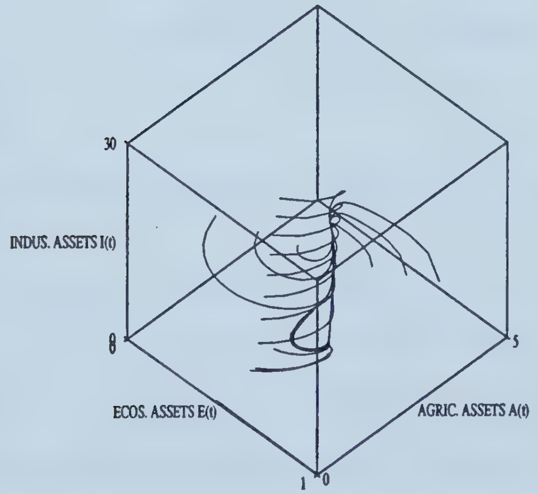


Figure 3.5: Phase portrait for system when $\gamma = 0.2$ and $\xi = 1.0$ and $\mu = 0$.

In Figure 3.5, we chose $\mu = 0$, $\gamma = 0.2$ and $\xi = 1$. In this case we have two axial steady states F_{1A} and F_{2A},which are locally unstable, one planar state F_{2P} lying in the interior of the (A-E)-plane, which is locally unstable because $\delta\alpha\vartheta > \xi(\alpha\kappa + \beta\vartheta)$

and one interior steady state F_{2I} since $\frac{\delta\alpha\vartheta}{\xi} > (\alpha\kappa+\beta\vartheta) > \frac{\vartheta\gamma\delta}{eta}$ and $(\beta\eta\vartheta+\kappa\gamma\xi) < \gamma\delta\vartheta$, which is locally stable. Figure 3.5 suggests that F_{2I} is also globally asymptotically stable.

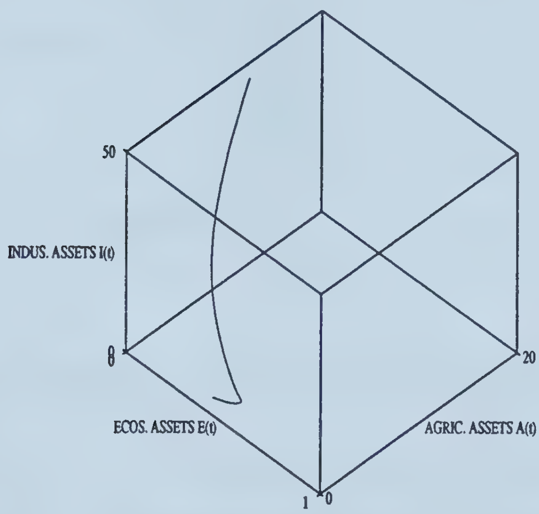


Figure 3.6: Phase portrait for system when $\gamma = 0.5$ and $\xi = 1.0$ and $\mu = 0$.

In Figure 3.6, we chose $\mu = 0$, $\gamma = 0.5$ and $\xi = 1.0$. In this case we have two axial steady states F_{1A} and F_{2A}, two planar steady states F_{1P} (since $\beta\eta < \gamma\delta$) lying in the interior of the (A-I)-plane and F_{2P} lying in the interior of the (A-E)-plane and no interior steady state since $\delta\alpha\vartheta > \xi(\alpha\kappa + \beta\vartheta)$ and $\eta(\alpha\kappa + \beta\vartheta) > \vartheta\gamma\delta$ but $(\beta\eta\vartheta + \kappa\gamma\xi) < \gamma\delta\vartheta$. In this case all the steady states are unstable, that is if one starts with a solution with positive initial conditions, the A and I components are unbounded.

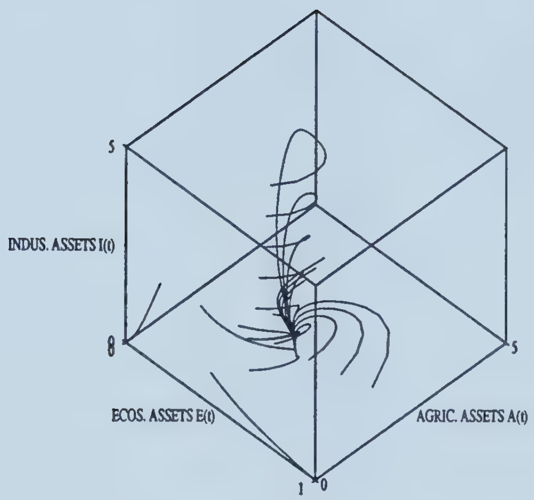


Figure 3.7: Phase portrait for system when $\gamma = 1.0$ and $\xi = 2.0$ and $\mu = 0$.

In Figure 3.7, we chose $\mu = 0$, $\gamma = 1.0$ and $\xi = 2.0$. In this case we have two axial steady states F_{1A} and F_{2A}, two planar steady states, F_{1P} (since $\beta\eta < \gamma\delta$) lying in the interior of the (A-I)-plane and F_{2P} lying in the interior of the (A-E)-plane and one interior steady state F_{2I} since $\frac{\delta\alpha\vartheta}{\xi} < (\alpha\kappa + \beta\vartheta) < \frac{\vartheta\gamma\delta}{eta}$. In this case all the steady states are locally unstable except F_{2I} which is locally stable. This figure also suggests that F_{2I} is globally stable.

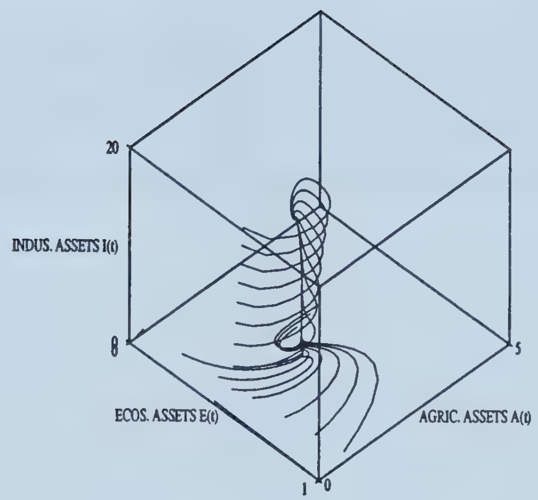


Figure 3.8: Phase portrait for system when $\gamma = 20/75$ and $\xi = 1.0$ and $\mu = 0$.

In Figure 3.8, we chose $\mu = 0$, $\gamma = 20/75$ and $\xi = 1.0$. In this case we have two axial steady states F_{1A} and F_{2A}, two planar steady states, F_{1P} (since $\beta\eta < \gamma\delta$)lying completely in the interior of the (A-I)-plane and F_{2P} lying in the interior of the (A-E)-plane and one interior steady state F_{2I} since $\frac{\delta\alpha\vartheta}{\xi} < (\alpha\kappa + \beta\vartheta) < \frac{\vartheta\gamma\delta}{\eta}$. In this case all the steady states are locally unstable except F_{2I} which is locally stable. This figure also suggests that F_{2I} is globally stable.

We observe from Figures 3.1 to 3.8 that any time the interior steady state exists, it seems to be globally asymptotically stable. We also observed that apart from Figure 3.6 (which is an uninteresting case) the steady state lying in the interior of the (A-E)-plane always appears to be globally asymptotically stable if the interior steady state does not exist. Thus we can always choose our model parameters such that all components of our system persist. The above Figures (i.e. Figures 3.1 to

3.8) illustrate the various cases which can occur if we decided not to correct the loss of ecospheric assets to agriculture but only depend on the natural recovery rate of ecospheric assets.

We now consider the case where there is a constant correction rate to the loss of ecospheric assets to agriculture (i.e. $\mu \neq 0$). The various cases are illustrated in Figures 3.9 to 3.18. We discuss each of these figures separately below.

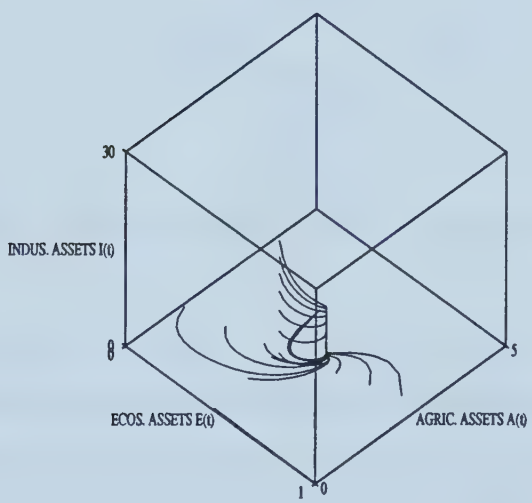


Figure 3.9: Phase portrait for system when $\gamma = 0.0$ and $\xi = 1.0$ and $\mu = 0.2$.

In Figure 3.9, we chose $\mu = 0.2$, $\gamma = 0.0$ and $\xi = 1.0$. In this case we have no axial study state (since $\mu \neq 0$), one planar steady state F_{3P} lying in the positive quadrant of the (A-E)-plane and one interior steady stateF_{3I} (since $E^* > \frac{\beta\xi}{\alpha\delta}$). The planar steady state is locally unstable while the interior steady is not only locally stable but appears to be globally asymptotically stable.

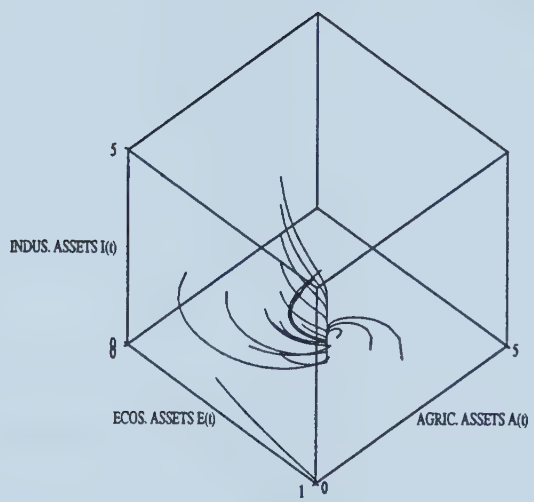


Figure 3.10: Phase portrait for system when $\gamma = 0.0$ and $\xi = 2.0$ and $\mu = 0.2$.

In Figure 3.10, we increase ξ from 1.0 to 2.0, (i.e. $\mu = 0$, $\gamma = 0.0$ and $\xi = 2.0$.). With this slight change, the interior steady state disappears since $E^* < \frac{\beta\xi}{\alpha\delta}$, leaving only the planar steady state in the interior of the (A-E)-plane. In this case the planar steady steady state is not only locally stable but appears to be globally asymptotically stable.

Figures 3.9 and 3.10 show that if the interaction between agriculture, industry and the ecosphere is such that there is a constant rate of correction to the loss of ecospheric assets to agriculture, and there is no net loss or gain of assets by agriculture to industry then starting with positive initial condition for the system, the system will persist if and only if $E^* > \frac{\beta\xi}{\alpha\delta}$ (that is if the constant rate of depreciation of industrial assets is small). If on the other hand the constant rate of depreciation of the industial assets is large so that $E^* < \frac{\beta\xi}{\alpha\delta}$, then only the industrial assets will go extinct.

In Figures 3.11 and 3.12, we consider the case where there is a net loss of assets by agriculture to industry while we assume a constant rate of correction to the loss of ecospheric assets.

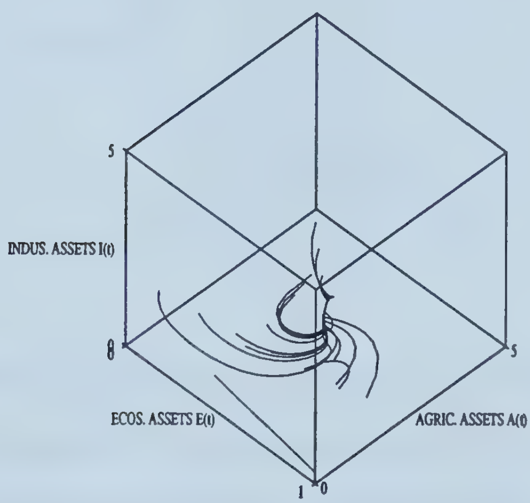


Figure 3.11: Phase portrait for system when $\gamma = -1.0$ and $\xi = 1.0$ and $\mu = 0.2$.

In figure 3.11, we choose $\mu = 0.2$, $\gamma = -1.0$ and $\xi = 1.0$. In this case the system has two steady states F_{4I}^{+} and F_{3P}. F_{3P} lies in the interior of the (A-E)-plane and is locally unstable whereas the interior steady state F_{4I}^{+} appears to be globally asymptotically stable.

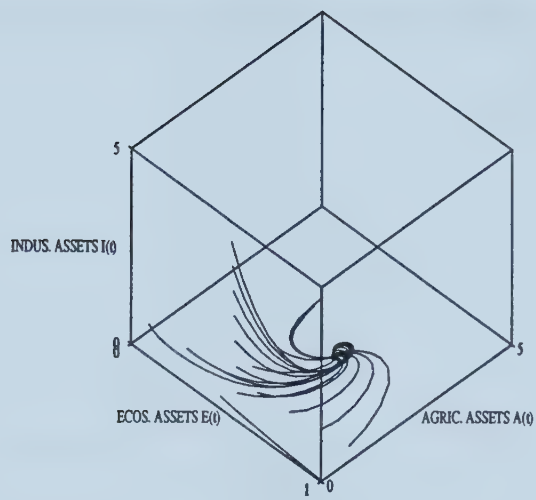


Figure 3.12: Phase portrait for system when $\gamma = -1.0$ and $\xi = 2.0$ and $\mu = 0.2$.

In Figure 3.12, we kept all the other parameters the same as in Figure 3.11, and increased the constant rate of depreciation of industry from 1.0 to 2.0. In this case the interior steady state disappears leaving only the planar state. The planar state now appears to be globally asymptoticaly stable. Thus Figures 3.11 and 3.12 are qualitatively the same as Figures 3.9 and 3.10.

Figures 3.11 and 3.12 show that if the interaction between agriculture, industry and the ecosphere is such that there is a constant rate of correction to the loss of ecospheric assets to agriculture, and there is a net loss of assets by agriculture to industry then starting with, positive initial conditions for the system, all components of the system will persist if and only if $E^* > \frac{\beta\xi}{\alpha\delta}$ (that is if the constant rate of depreciation of industrial assets is small). If on the other hand the constant rate of depreciation of the industial assets is large such that $E^* < \frac{\beta\xi}{\alpha\delta}$, then the industrial assets will go extinct.

In Figures 3.13 and 3.14, we assume that there is a net gain in assets by agriculture fron industry but the gain is relatively small so that $\beta\eta > \gamma\delta$. We also assume there is a constant rate of correction to the loss of ecospheric assets to agriculture.

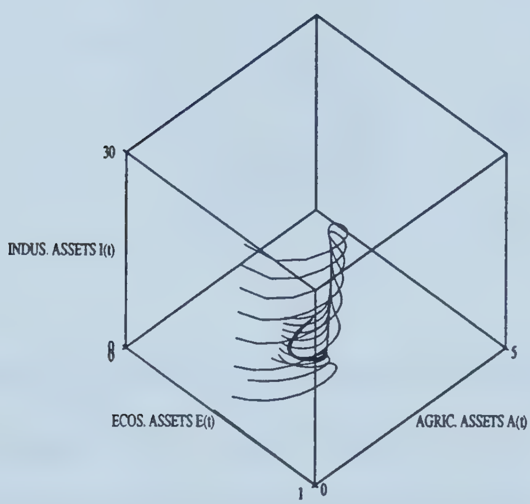


Figure 3.13: Phase portrait for system when $\gamma = 0.2$ and $\xi = 1.0$ and $\mu = 0.2$.

In Figure 3.13, we chose $\mu = 0.2$, $\gamma = 0.2$ and $\xi = 1.0$. We observe that this figure is qualitatively the same as that of Figure 3.11 and 3.9 (i.e. we have two steady states: F_{3P} in the interior of (A-E)-plane which is locally unstable and F_{4I}^{+} which appears to be globally asymptotically stable).

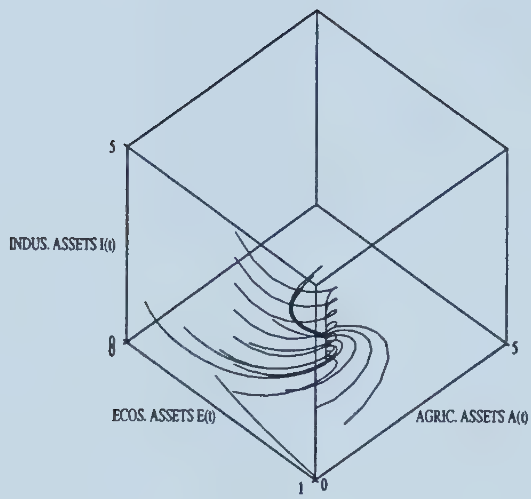


Figure 3.14: Phase portrait for system when $\gamma = 0.2$ and $\xi = 2.0$ and $\mu = 0.2$.

In Figure 3.14, we increase ξ from 1.0 to 2.0 (i.e. $\mu = 0.2$, $\gamma = 0.2$ and $\xi = 2.0$), in this case we have only the planar steady state in the interior of the (A-E)-plane and it appears to be globally asymptotically stable.

Figures 3.9 to 3.14 indicate that if there is a net loss, no net loss or gain or a net gain (but relatively small such that $\beta\eta > \gamma\delta$) by agriculture as a result of the interaction between agriculture, industry and the ecosphere, then the interior steady state (if it exists) appears to be always globally asymptotically stable. On the other hand if the interior steady states does not exist, then the only steady state we have is the one in the interior of the (A-E)-plane and that appears to be always globally asymptotically stable.

In Figures 3.15, 3.16, 3.17 and 3.18, we assume that there is a net gain in assets by agriculture and that this gain is relatively large so that $\beta\eta < \gamma\delta$. In fact in Figures 3.15 and 3.16, we even assumed that the gain is so large such that $(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha < 0$.

We also assumed there is a constant correction rate for the loss of assets by the ecosphere.

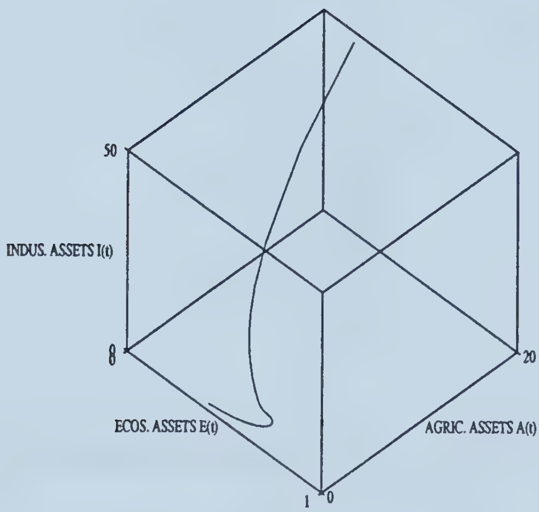


Figure 3.15: Phase portrait for system when $\gamma = 1.0$ and $\xi = 1.0$ and $\mu = 0.2$.

In Figure 3.15 we chose $\mu = 0.2$, $\gamma = 1.0$ and $\xi = 1.0$. For this case we have no interior steady state and the only planar steady state F_{3P} that we have is locally unstable. Hence if we start with postive initial conditions, the solutions grow unbounded in both the I and A directions. But if we increase the constant depreciation rate of industry from 1.0 to 2.0, then we have an interior steady state given by F_{4I}^{-} and a planar state given by F_{3P}. This is shown in Figure 3.16 (in this case we chose $\mu = 0.2$, $\gamma = 1.0$ and $\xi = 2.0$). The interior steady state appears to be globally asymptotically stable.

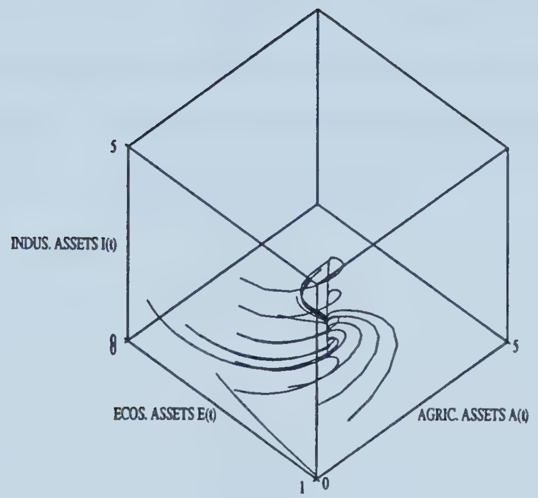


Figure 3.16: Phase portrait for system when $\gamma = 1.0$ and $\xi = 2.0$ and $\mu = 0.2$.

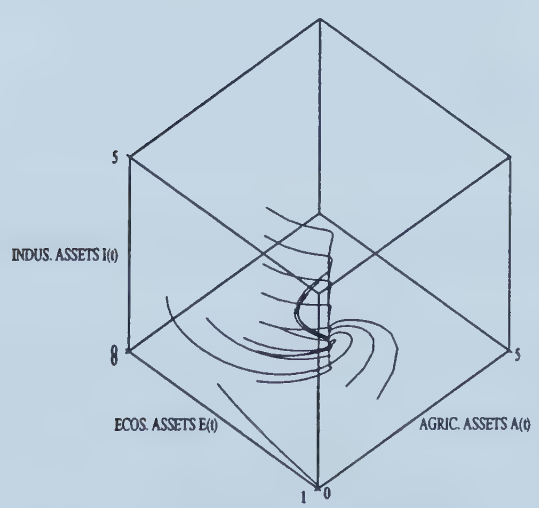


Figure 3.17: Phase portrait for system when $\gamma = 0.5$ and $\xi = 2.0$ and $\mu = 0.2$.

In Figures 3.17 and 3.18, we reduced the value of γ so that $\beta\eta < \gamma\delta$ but $(\beta\eta - \gamma\delta)\vartheta + \kappa\eta\alpha > 0$.
In Figure 3.17, we chose $\mu = 0.2$, $\gamma = 0.5$ and $\xi = 2.0$. In that case we have two steady states F_{4I}^{-} and F_{3P}. F_{3P} is locally unstable and F_{4I}^{-} appears to be globally stable.
In Figure 3.18, we reduced the value of ξ from 2.0 to 1.6 while keeping all the other parameters as in Figure 3.17. We observe that in this case we have only one steady state lying in the interior of the positive (A-E)-plane. This steady state is locally unstable.

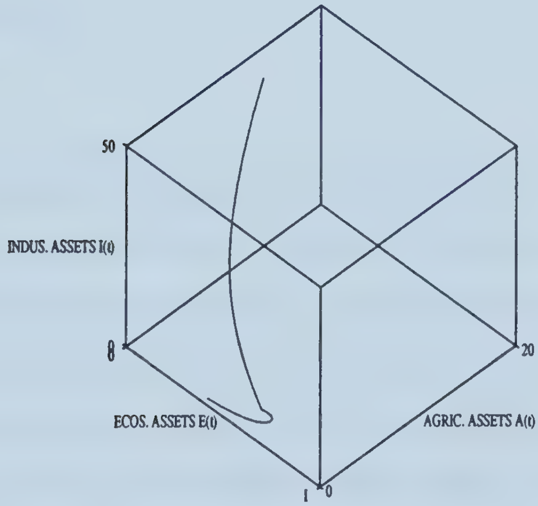


Figure 3.18: Phase portrait for system when $\gamma = 0.5$ and $\xi = 1.6$ and $\mu = 0.2$.

Chapter 4

Ecospheric Recovery Model with Hysteresis

In Chapter 3, we studied the ecospheric recovery model of an agricultural-industrial system by assuming that the effort f(E,A) was a constant, μ. This constant was referred to as the constant correction rate. We observed that for a given agricultural asset, this effort will initially cause the ecospheric asset to grow until it attains a certain maximum value at a certain time T, E_{max}=E(T), say. After this time T, there will be no growth in the ecospheric assets even if we continue to put in this effort, μ (see Figures 4.1, 4.2 and 4.3). Hence after this time T, farmers may either decide to stop putting in this effort to replenish the environment (since there is no effect of this effort on the ecosphere after this time) or reduce their effort. They do so because there is a cost associated with putting in effort. This forms the basis of our model in this chapter. Thus in this chapter, we consider the ecospheric model studied in Chapter 2 modified by the addition of a hysteresis term $f[E(\cdot)](t)$ in the equation for the ecosphere.

Figure 4.1: (mu=0)

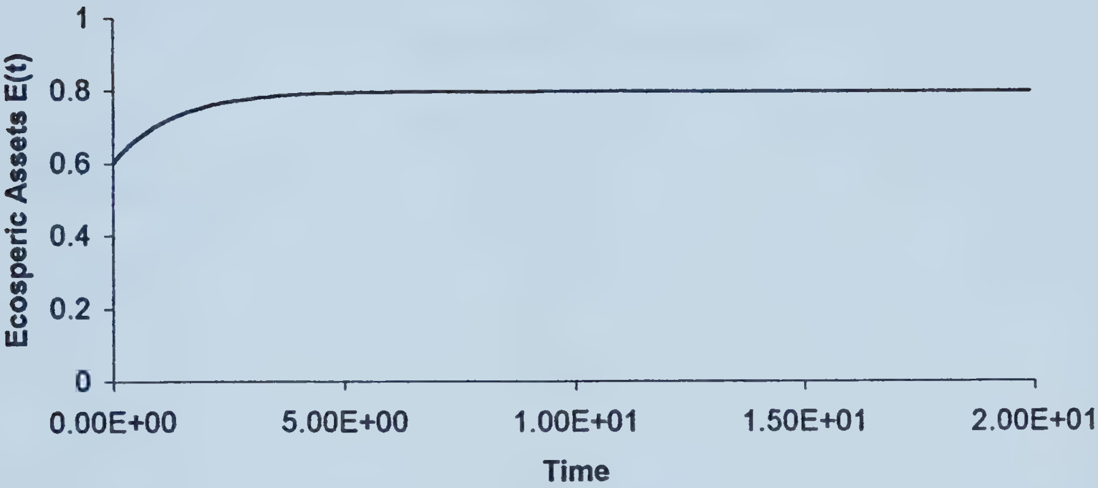


Figure 4. 2 : (mu=0.2)

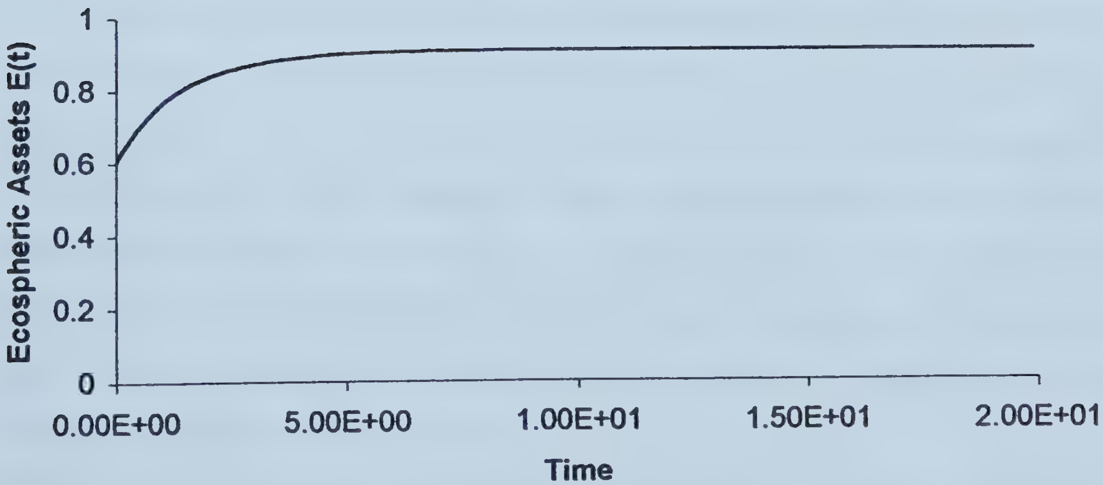
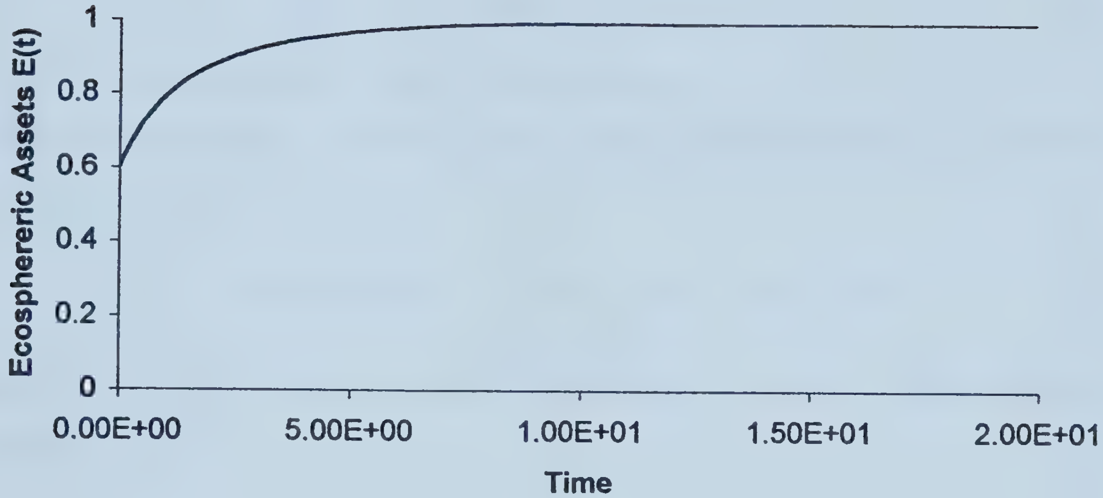


Figure 4.3: (mu=0.3)



We will initially discuss only the ecospheric equation in the recovery model by discussing the various modifications made and the reason behind those modifications. We will then solve these modified equations numerically for a given agricultural asset. This is done by writing a Fortran program that solves the corresponding differential equations using a fourth order Runge-Kutta method. Four different cases of these modified equations will be considered. They are (i) the switch case (ii) the simple relay case (iii) the stop case and (iv) the modified stop case. Later in this chapter we will couple these modified equations with the unmodified agricultural and industrial equations in the recovery model and solve these numerically. These results will then be compared with our previous results.

We recall that the recovery model studied in Chapter 3 was given by the following system of differential equations

$$\frac{dA}{dt} = \alpha EA - \beta A^2 + \gamma AI, \tag{4.1}$$

$$\frac{dI}{dt} = -\xi I - \eta I^2 + \delta AI, \tag{4.2}$$

$$\frac{dE}{dt} = -\kappa EA + \vartheta(1-E)E + f(E,A), \tag{4.3}$$

with $f(E,A) = \mu$, $E(0) \geq 0$, $I(0) \geq 0$, and $A(0) \geq 0$.

In the next four sections, we will be solving the modified ecospheric equation with the following parameter values

$$\kappa = 0.5, \vartheta = 2.0, \mu = 0.2, A_0 = A(0) = A(t) = 0.8.$$

Thus we will be assuming that A(t) is a constant when solving the modified ecospheric equations.

4.1 Switch Case

In this section we treat the modified ecospheric equation with a switch. Here we consider farmers who have a maximum threshold E_{max} for the ecosphere. If their ecospheric assets fall below E_{max}, they put in an effort μ (which is a constant) to raise their ecospheric assets, and if it grows beyond E_{max}, they turn off their effort. Thus their effort is a function of the ecospheric assets at anytime and is given by

$$f(E,A) = f[E(\cdot)](t) = \begin{cases} \mu & \text{if } 0 \leq E(t) < \mathrm{E}_{max} \\ 0 & \text{if } \mathrm{E}_{max} \leq E(t) \leq 1 \end{cases}.$$

Hence the modified ecospheric equation in this case is given by

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E + \mu & \text{if } 0 \leq E(t) < \mathrm{E}_{max} \\ -\kappa EA + \vartheta(1-E)E & \text{if } \mathrm{E}_{max} \leq E(t) \leq 1 \end{cases}. \tag{4.4}$$

with $E(0) \geq 0$ and $0 \leq E(t) \leq 1$.

Figure 4.4: (mu=0.2, Emax=0.88, Eo=0.6)

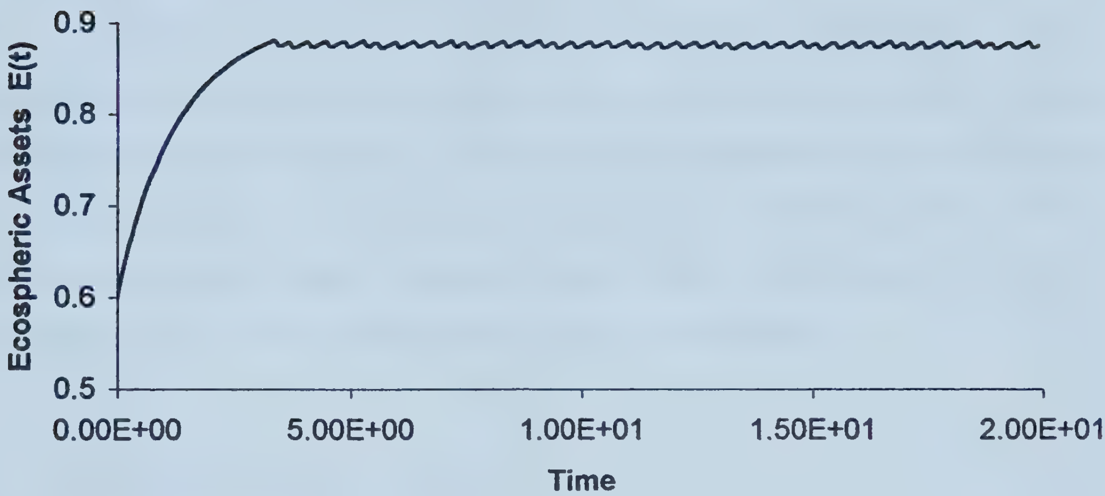
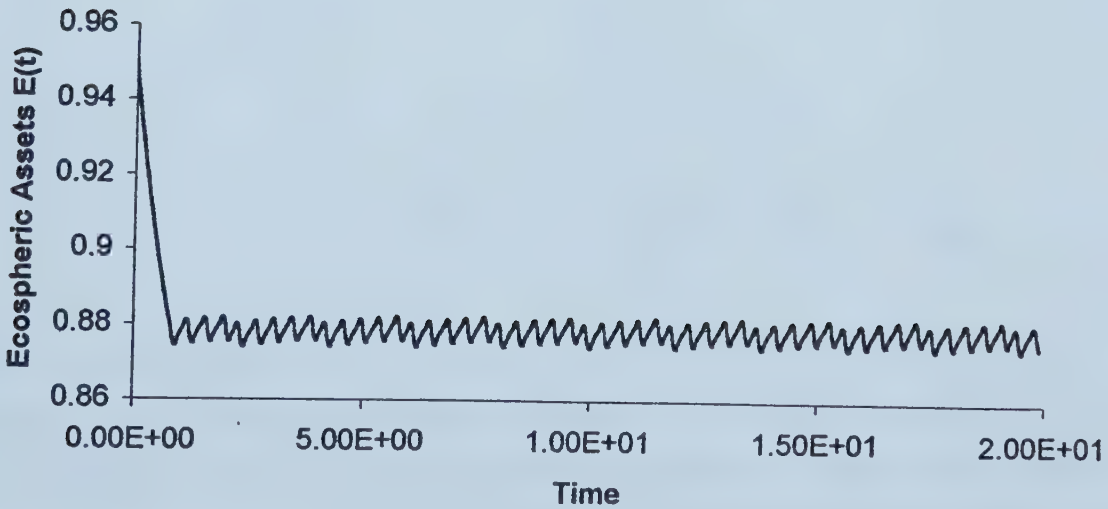


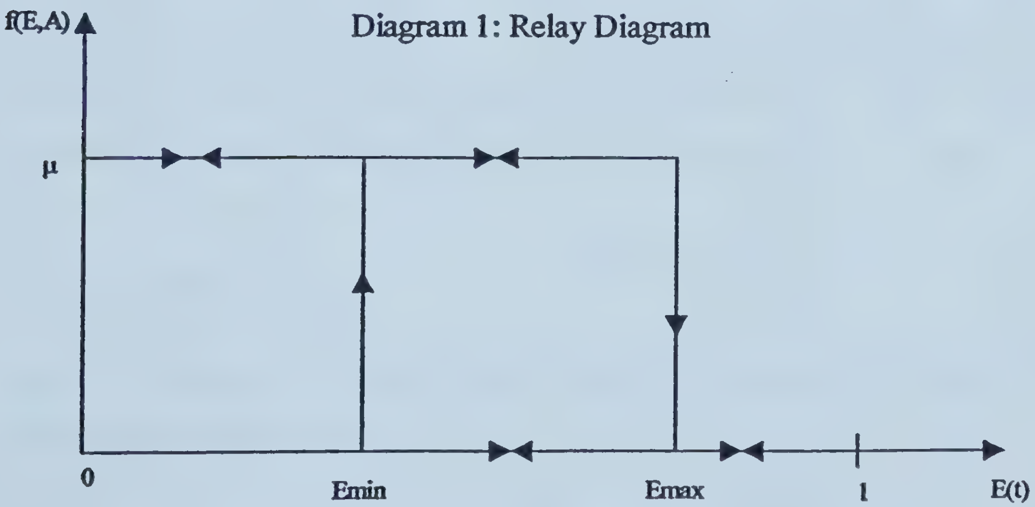
Figure 4. 5: (mu=0.2,Emax=0.88,Eo=0.95)



We note that the threshold E_{max} is only significant if it lies between the maximum possible levels of ecospheric assets attained when there is no correction and when there is a constant correction rate, μ. The solution to Equation (4.4) for a fixed agricultural asset is given in Figures 4.4 and 4.5. In Figure 4.4, $E_0 = E(0) < E_{max}$ and in Figure 4.5, $E_0 = E(0) > E_{max}$. In Figure 4.4, the assets grow until E attains E_{max}, and E(t) oscillates around this value as times goes on. Similarly in Figure 4.5, the assets decline until E reaches E_{max} and then oscillates rapidly around this value as time goes on, thus making this system unstable asymptotically.

4.2 Simple Relay Case

Diagram 1: Relay Diagram



In this section we consider the modified ecospheric equation with a simple relay. In this case, we are dealing with farmers who have two thresholds E_{max} and E_{min} for their ecospheric assets. E_{min} is the minimum acceptable level of asset, so they always try to maintain their assets above this level. E_{max} is the maximum level of assets they wish to attain. Once again we note that these thresholds are only significant if they

lie between the maximum possible levels of ecospheric assets attained when there is no correction and when there is a constant rate of correction μ.

If these farmers start with an ecospheric asset at a level less than E_{max} then they put in an effort μ so as to raise the level of assets to E_{max}. After attaining E_{max}, they turn off their effort and this may cause the level of assets to decline. Since E_{min} is the minimum acceptable level of ecospheric assets, they do nothing about the decline in assets until it gets to E_{min}. When the assets get to E_{min}, they put in an effort μ again so as to raise their ecospheric assets to E_{max}. This process is repeated over and over again. Similarly, if they start with assets higher than E_{max}, then they do nothing about it until it falls below E_{min}. At this time they put in an effort μ so as to raise the level of assets to E_{max}. When the assets get to E_{max} they turn off their effort. This process is repeated indefinitely. Thus the effort f(E,A) at time t is a function of the ecospheric asset at time t and the last threshold passed. In this case we have two subcases to consider. These cases are $E(0) < E_{max}$ and $E(0) \geq E_{max}$.

4.2.1 $E(0) < E_{max}$

Let T_k , k=0,1,2,3, ... be the time at which E(t) passes the k^{th} threshold, $T_0 = 0$. Define the k^{th} threshold by

$$k^{th} \quad \text{threshold} = \begin{cases} E_{max} & \text{if } k = 2n-1,\ n \epsilon Z^+ \\ E_{min} & \text{if } k = 2n,\ n \epsilon Z^+ \end{cases} .$$

Hence the effort f(E,A) in this case is given by

$$f(E,A) = f[E(\cdot)](t) = \begin{cases} \mu & \text{if } T_{2n} \leq t < T_{2n+1} \\ 0 & \text{if } T_{2n+1} \leq t < T_{2n+2},\ n = 0,1,2,3,... \end{cases} .$$

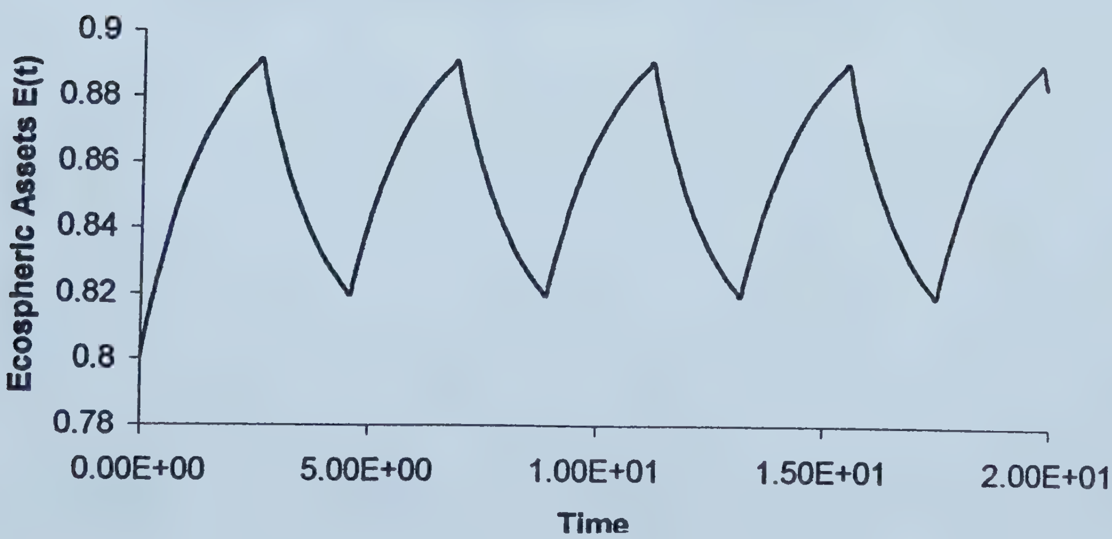
Hence the modified ecospheric equation in this case is given by

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E + \mu & \text{if } T_{2n} \leq t < T_{2n+1},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E & \text{if } T_{2n+1} \leq t < T_{2n+2},\ n = 0,1,2,3,... \end{cases} \quad . \tag{4.5}$$

with $E(0) \geq 0$ and $0 \leq E(t) \leq 1$.

We solve Equation (4.5) numerically and the output is as shown in Figure 4.6. Initially the assets grow until E(t) attains E_{max}, at this point the effort is turned off, hence the assets begin to decline until E(t) attains E_{min}, at which time the effort is turn on again so the assets begin to grow.Thus we have a periodic solution with a maximum value at E_{max} and a minimum value at E_{min}.

Figure 4. 6: (Emax=0.89,Emin=0.82,Eo=0.8,mu=0.2)



4.2.2 $E(0) \geq \mathrm{E}_{max}$

As in the above subsection, we will let T_k , k=0,1,2,3, ... be the time at which E(t) passes the k$^{\text{th}}$ threshold, $T_0 = 0$. Define the k$^{\text{th}}$ threshold by

$$\text{k}^{\text{th}} \quad \text{threshold} = \begin{cases} \mathrm{E}_{min} & \text{if } k = 2n-1,\ n\epsilon Z^{+} \\ \mathrm{E}_{max} & \text{if } k = 2n,\ n\epsilon Z^{+} \end{cases}.$$

Hence the effort f(E,A) is given in this case is given by

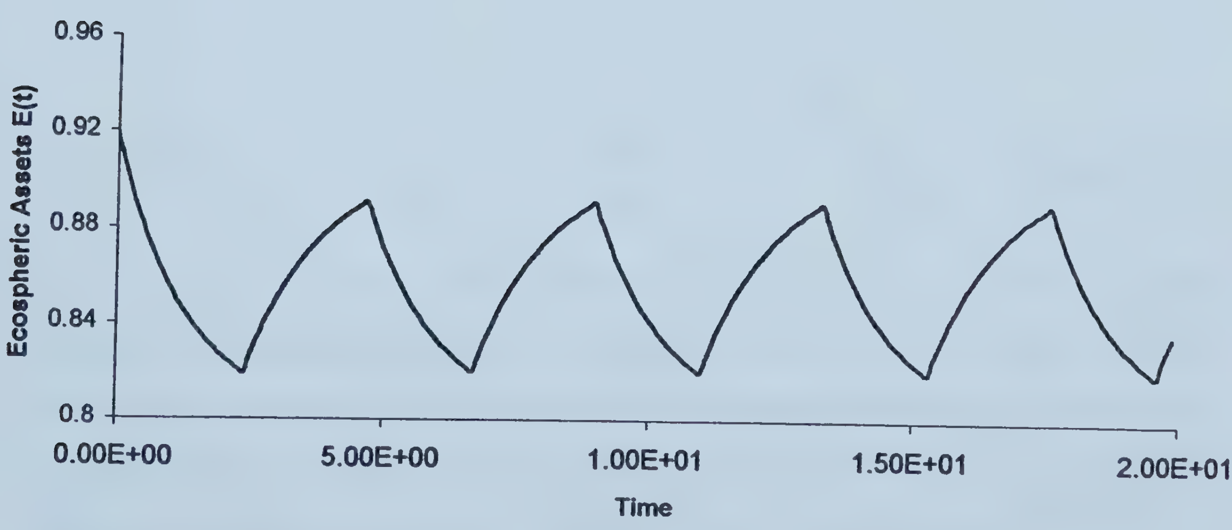
$$f(E,A) = f[E(\cdot)](t) = \begin{cases} 0 & \text{if } T_{2n} \leq t < T_{2n+1} \\ \mu & \text{if } T_{2n+1} \leq t < T_{2n+2},\ n = 0,1,2,3,... \end{cases}.$$

Hence the modified ecospheric equation in this case is given by

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E & \text{if } T_{2n} \leq t < T_{2n+1},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \mu & \text{if } T_{2n+1} \leq t < T_{2n+2},\ n = 0,1,2,3,... \end{cases}. \tag{4.6}$$

with $E(0) \geq 0$ and $0 \leq E(t) \leq 1$.

Figure 4.7: (Emax=0.89,Emin=0.82, Eo=0.92, mu=0.2)

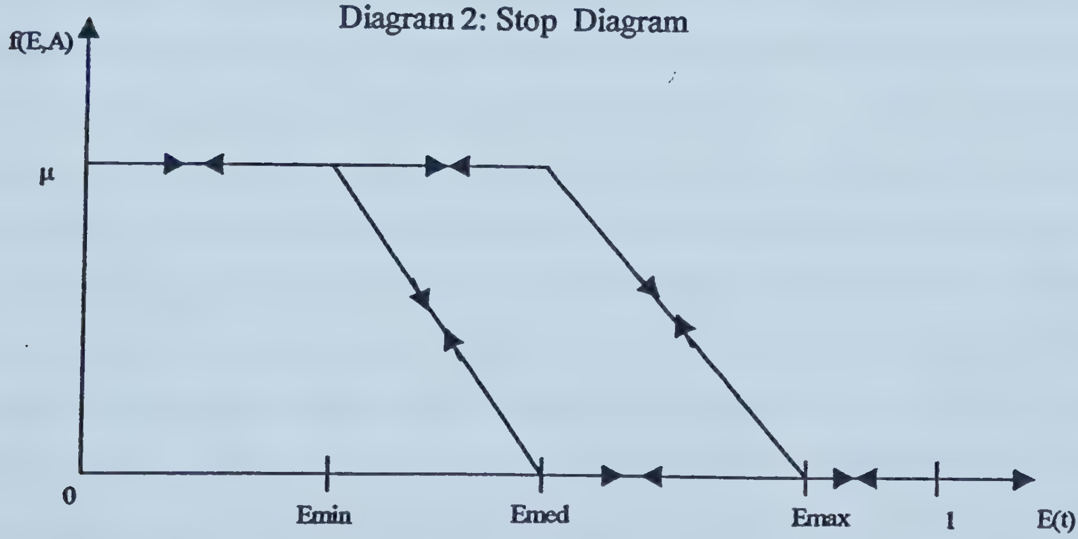


We solve Equation (4.6) numerically and the output is as shown in Figure 4.7. In this case the assets initially decline until E(t) attains E_{min} and at this point E(t) starts to grow because the effort is turn on. It grows until E(t) attains E_{max} at which time E(t) begins to decline, thus giving us a periodic solution with a maximum value at E_{max} and a minimum value at E_{min}.

The graph describing f(E,A) in Section 4.2 is as shown in Diagram 1 and the flow chart describing the numerical solution of Equations (1.5) and (1.6) is as shown in Chart 1.

4.3 Stop Case

Diagram 2: Stop Diagram



In this section we treat the modified ecospheric equation with a stop, that is, we will be dealing with farmers who have three different thresholds, E_{max}, E_{med}, E_{min} for their ecospheric assets. E_{min} is the minimum acceptable level of ecospheric assets. E_{max} is the maximum level of ecospheric asset they wish to obtain. E_{med} is a threshold which lies between E_{min} and E_{max}. If E(t) falls below E_{med}, farmers are forced to put

in more or increase their effort so that their assets will not fall below E_{min} , whereas if E(t) grows beyond E_{med}, then they naturally reduce their effort. In general therefore, farmer will always want to maintain their assets level between E_{med} and E_{max}.

In general if these farmers start with an ecospheric asset at a level less than E_{med}, then they put in a constant effort μ so as to raise the level to E_{med}, after attaining E_{med}, they reduce their effort and put in an effort which is proportional to the ecospheric level at that time (that is if $E_{med} < E(t) \leq E_{max}$). If this effort raises the level of ecophereric assets to E_{max}, then they turn off their effort completely. This will in general cause the ecospheric assets to decline over time, but these farmers do nothing about this decline until it reaches E_{med}. At this time they put in an effort proportional to the ecospheric assets at that time and if this is not high enough, the assets may continue to decline. If this is the case, they will keep on putting in this effort until it declines to E_{min}, after which they increase their effort to μ, a constant. It is hoped that this will raise the level of assets over time to E_{med}. This whole process is repeated indefinitely. On the other hand, if the initial ecospheric asset is less than E_{max} but greater than E_{med}, then they will put in an effort which is proportional to the level of assets until it reaches E_{max}, at which time they turn off their effort and follow the same routine as described above. Similarly if they start with an asset at a level higher that E_{max}, then they don't put in any effort until the asset falls below E_{med} at which time they put in an effort which is proportional to the asset level and follow the same routine as describe above. Just as described in Section 4.2, the effort f(E,A) is a function of the ecospheric assets and also the last passed threshold.

We will therefore consider the effort f(E,A) in this case under three subcases (i) $E(0) < E_{med}$, (ii) $E_{med} \leq E(0) < E_{max}$ and (iii) $E_{max} \leq E(0) \leq 1$. Let T_k, $k = 0, 1, 2, 3, ...$ be the time at which E(t) passes the k$^{\text{th}}$ threshold, with $T_0 = 0$

4.3.1 $0 \leq E(0) < \mathrm{E}_{med}$

We define the $\mathrm{k^{th}}$ threshold by

$$\mathrm{k^{th}} \quad \text{threshold} = \begin{cases} \mathrm{E}_{med} & \text{if } k = 4n+1,\ n = 0,1,2,3... \\ \mathrm{E}_{max} & \text{if } k = 4n+2,\ n = 0,1,2,3.... \\ \mathrm{E}_{med} & \text{if } k = 4n+3,\ n = 0,1,2,3,... \\ \mathrm{E}_{min} & \text{if } k = 4n+4,\ n = 0,1,2,3,... \end{cases}.$$

The effort f(E,A) is given by

$$f(E,A) = f[E(\cdot)](t) = \begin{cases} \mu & \text{if } T_{4n} \leq t < T_{4n+1},\ n = 0,1,2,3,... \\ \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{med}} & \text{if } T_{4n+1} \leq t < T_{4n+2},\ n = 0,1,2,3,..., \\ 0 & \text{if } T_{4n+2} \leq t < T_{4n+3},\ n = 0,1,2,3,... \\ \frac{\mu(\mathrm{E}_{med}-E)}{\mathrm{E}_{med}-\mathrm{E}_{min}} & \text{if } T_{4n+3} \leq t < T_{4n+4},\ n = 0,1,2,3,... \end{cases}.$$

The above give rise to the following modified ecospheric equation

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E + \mu & \text{if } T_{4n} \leq t < T_{4n+1},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{med}} & \text{if } T_{4n+1} \leq t < T_{4n+2},\ n = 0,1,2,3,..., \\ -\kappa EA + \vartheta(1-E)E & \text{if } T_{4n+2} \leq t < T_{4n+3},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \frac{\mu(\mathrm{E}_{med}-E)}{\mathrm{E}_{med}-\mathrm{E}_{min}} & \text{if } T_{4n+3} \leq t < T_{4n+4},\ n = 0,1,2,3,... \end{cases}. \tag{4.7}$$

with $E(0) \geq 0$ and $0 \leq E(t) \leq 1$.

The solution to Equation (4.7) is as shown in Figures 4.8 and 4.9. The asset grows almost linearly from $E_0 = E(0)$ to E_{med}. Then it will keep on growing from E_{med} until it attains a certain critical value $\bar{E}$ in between E_{med} and E_{max}. At this point the net gain in assets is equal to the net degradation of the ecosphere so the assets remain constant. In fact if at this point one should slightly increase his effort the asset would grow, and if on the other hand one decreases his effort the asset would decline. It

should be noted that this critical value $\bar{E}$ is good enough since it lies between E_{med} and E_{max}.

Figure 4.8:
(Emax=0.89, Emed=0.84, Emin=0.82, Eo=0.79, mu=0.2)

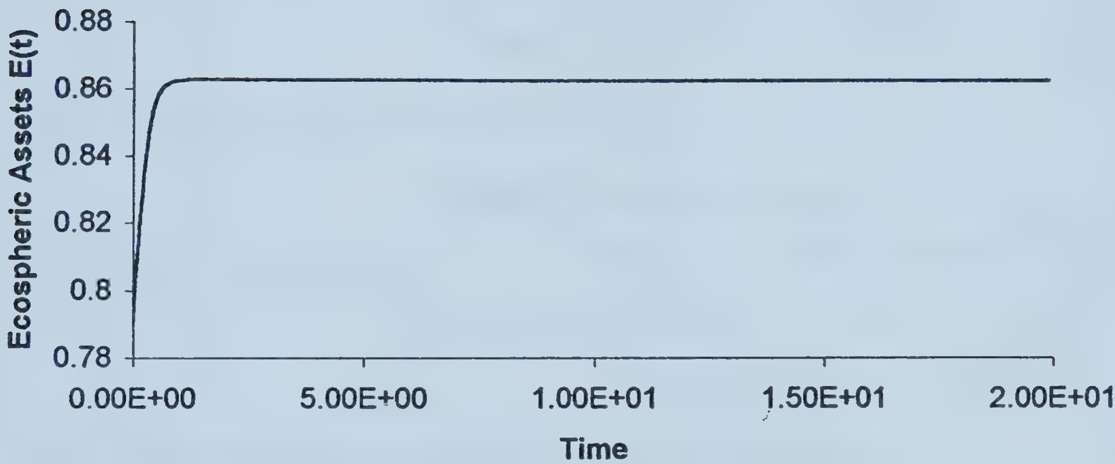
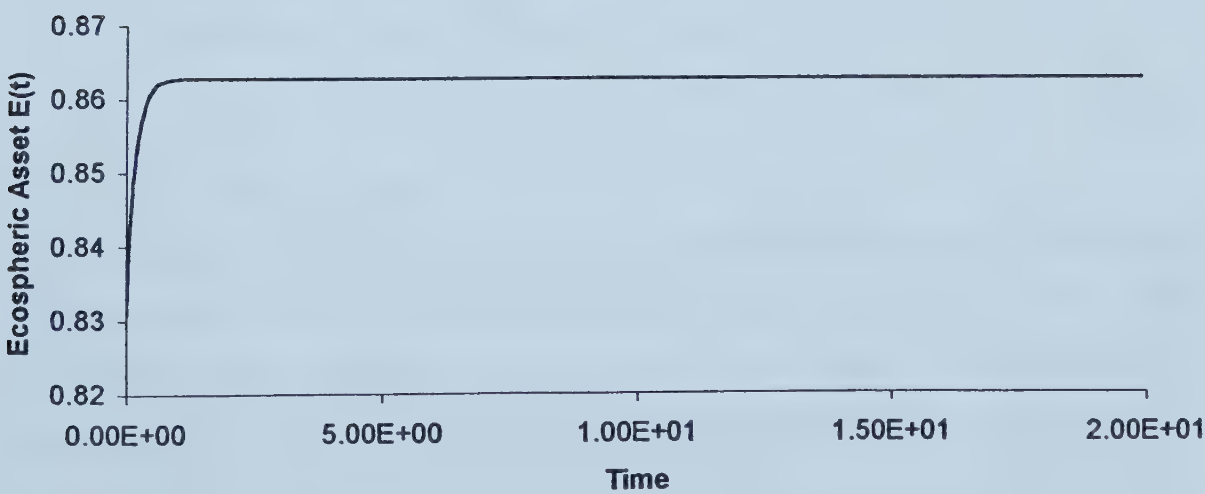


Figure 4. 9:
(Emax=0.89,Emed=0.84,Eo=0.82,Eo=0.83,mu=0.2)



4.3.2 $\mathrm{E}_{med} \leq E(0) < \mathrm{E}_{max}$

In this subsection we define the k^{th} threshold as follows

$$\mathrm{k}^{\mathrm{th}} \quad \text{threshold} = \begin{cases} \mathrm{E}_{max} & \text{if } k = 4n+1,\ n = 0,1,2,3... \\ \mathrm{E}_{med} & \text{if } k = 4n+2,\ n = 0,1,2,3.... \\ \mathrm{E}_{min} & \text{if } k = 4n+3,\ n = 0,1,2,3,... \\ \mathrm{E}_{med} & \text{if } k = 4n+4,\ n = 0,1,2,3,... \end{cases}.$$

In this case the effort f(E,A) is defined as follows

$$f(E,A) = f[E(\cdot)](t) = \begin{cases} \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{med}} & \text{if } T_{4n} \leq t < T_{4n+1},\ n = 0,1,2,3,..., \\ 0 & \text{if } T_{4n+1} \leq t < T_{4n+2},\ n = 0,1,2,3,... \\ \frac{\mu(\mathrm{E}_{med}-E)}{\mathrm{E}_{med}-\mathrm{E}_{min}} & \text{if } T_{4n+2} \leq t < T_{4n+3},\ n = 0,1,2,3,... \\ \mu & \text{if } T_{4n+3} \leq t < T_{4n+4},\ n = 0,1,2,3,... \end{cases}.$$

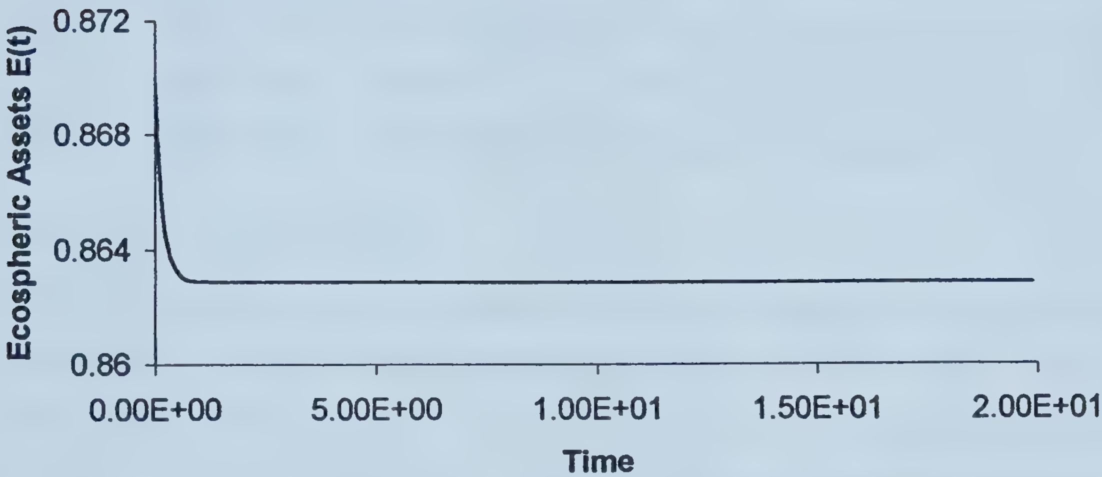
This then gives rise to the following modified ecospheric equation

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E + \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{med}} & \text{if } T_{4n} \leq t < T_{4n+1},\ n = 0,1,2,3,..., \\ -\kappa EA + \vartheta(1-E)E & \text{if } T_{4n+1} \leq t < T_{4n+2},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \frac{\mu(\mathrm{E}_{med}-E)}{\mathrm{E}_{med}-\mathrm{E}_{min}} & \text{if } T_{4n+2} \leq t < T_{4n+3},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \mu & \text{if } T_{4n+3} \leq t < T_{4n+4},\ n = 0,1,2,3,... \end{cases}. \tag{4.8}$$

with $E(0) \geq 0$ and $0 \leq E(t) \leq 1$.

The numerical solution to Equation (4.8) is as shown in Figure 4.10. In this figure $E(0)$ is chosen higher than $\bar{E}$, hence the assets decline from $E(0)$ to $\bar{E}$, after which it remains constant since there is no net gain or loss in assets. If however, $E(0)$ is chosen such that $E(0) < \bar{E}$, then the assets will grow initially before stabilizing at $\bar{E}$.

Figure 4.10:
(Emax=0.89,Emed=0.84,Emin=0.82,Eo=0.87,mu=0.2)



4.3.3 $\mathrm{E}_{max} \leq E(0) \leq 1$

In this case we define the k$^{\text{th}}$ threshold as follows

$$\mathrm{k^{th}} \quad \text{threshold} = \begin{cases} \mathrm{E}_{med} & \text{if } k = 4n+1,\ n = 0,1,2,3... \\ \mathrm{E}_{min} & \text{if } k = 4n+2,\ n = 0,1,2,3.... \\ \mathrm{E}_{med} & \text{if } k = 4n+3,\ n = 0,1,2,3,... \\ \mathrm{E}_{max} & \text{if } k = 4n+4,\ n = 0,1,2,3,... \end{cases}.$$

In the case the effort f(E,A) is defined by

$$f(E,A) = f[E(\cdot)](t) = \begin{cases} 0 & \text{if } T_{4n} \leq t < T_{4n+1},\ n = 0,1,2,3,... \\ \frac{\mu(\mathrm{E}_{med}-E)}{\mathrm{E}_{med}-\mathrm{E}_{min}} & \text{if } T_{4n+1} \leq t < T_{4n+2},\ n = 0,1,2,3,... \\ \mu & \text{if } T_{4n+2} \leq t < T_{4n+3},\ n = 0,1,2,3,... \\ \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{med}} & \text{if } T_{4n+3} \leq t < T_{4n+4},\ n = 0,1,2,3,..., \end{cases}.$$

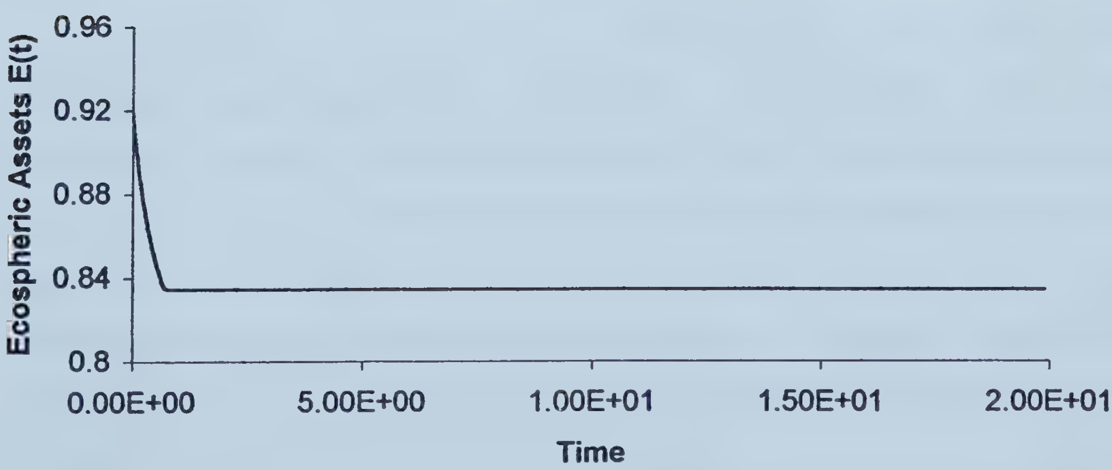
Hence the modified ecospheric equation in this case is given by

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E & \text{if } T_{4n} \leq t < T_{4n+1},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \frac{\mu(\mathrm{E}_{med}-E)}{\mathrm{E}_{med}-\mathrm{E}_{min}} & \text{if } T_{4n+1} \leq t < T_{4n+2},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \mu & \text{if } T_{4n+2} \leq t < T_{4n+3},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{med}} & \text{if } T_{4n+3} \leq t < T_{4n+4},\ n = 0,1,2,3,..., \end{cases} \tag{4.9}$$

with $E(0) \geq 0$ and $0 \leq E(t) \leq 1$.

The numerical solution to the Equation (4.9) above is as shown in Figure 4.11. Since $E(0) > \mathrm{E}_{max}$, no effort is made to replenish the loss in ecospheric assets. This causes the ecospheric assets to decline until it attains E_{med}. At this point a little effort is made to replenish the ecosphere. However this is not good enough, so the assets continue to decline. This decline continues until it reaches $\underline{E}$ where $\mathrm{E}_{min} < \underline{E} < \mathrm{E}_{med}$. At this point the effort is just good enough to maintain the assets at this level.

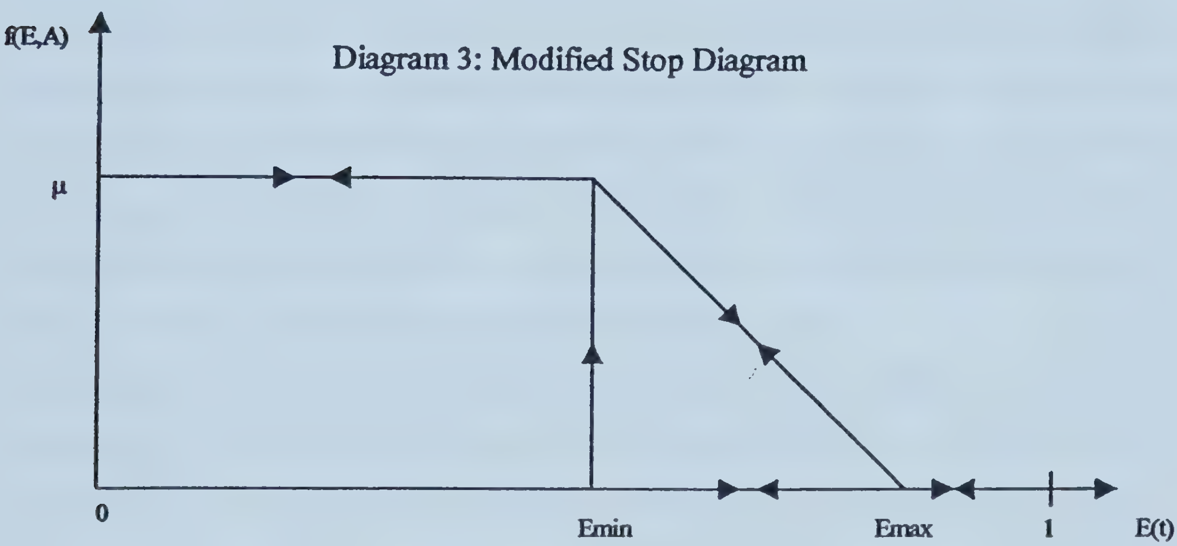
Figure 4.11:
(Emax=0.89, Emed=0.84,Emin=0.82,Eo=0.92,mu=0.2)



The graph describing f(E,A) in Section 4.3 is as shown in Diagram 2. The flow chart describing the solution to the various ecospheric equations described in Section 4.3 is as shown in Chart 2.

4.4 Modified Stop case

Diagram 3: Modified Stop Diagram



As shown in Figure 4.11, there is the possibility that (i.e. for the stop case) if the assets grow beyond E_{max}, then that method can cause the ecospheric assets to stabilize between E_{min} and E_{med}, which is not required. Thus if the assets decline to E_{med}, instead of putting in an effort that is proportional to the ecospheric level at that time, we will put in a full effort of μ. This gives rise to the modified Ecospheric equation with a modified stop. In this case we have two thresholds instead of three, E_{med} and E_{max}. As in the case of simple relay, we will refer to these thresholds as E_{min} and E_{max}.

Here we are dealing with farmers with the following behaviour. If these farmers start

with ecospheric assets at a level lower than E_{min}, then they put in an effort of μ so as to raise the level of these assets. If these assets rise to E_{min}, they reduce their effort depending on the level of the ecopheric assets.This may cause their assets to still grow (if the reduction is done properly) until it attains E_{max}, at which time they turn off their effort completely. Generally this action will cause their assets level to decline but they do not react to this decline until their assets falls below E_{min} at which time they put in an effort of μ again and follow the same pattern described above. On the other hand if they start with initial assets at a level higher than E_{min} but lower than E_{max}, then they put in an effort proportional to the level of assets until the assets grow to E_{max}. At this time they turn off their effort, causing the assets to decline. If the assets fall below E_{min}, they put in an effort of μ and follow the same pattern described above. A similar process is followed if $E(0) > \mathrm{E}_{max}$.

As in the previous cases, we will consider the effort f(E,A) in this case under three subcases, (i) $E(0) < \mathrm{E}_{min}$, (ii) $\mathrm{E}_{min} \leq E(0) < \mathrm{E}_{max}$ and (iii) $\mathrm{E}_{max} \leq E(0) \leq 1$. Let T_k, k=0,1,2,3,... be the time at which E(t) passes the k^{th} threshold, with $T_0 = 0$.

4.4.1 $0 \leq E(0) < \mathrm{E}_{min}$

We define the k^{th} threshold as

$$\mathrm{k}^{\mathrm{th}} \quad \text{threshold} = \begin{cases} \mathrm{E}_{min} & \text{if } k = 3n+1,\ n = 0,1,2,3... \\ \mathrm{E}_{max} & \text{if } k = 3n+2,\ n = 0,1,2,3.... \\ \mathrm{E}_{min} & \text{if } k = 3n+3,\ n = 0,1,2,3,... \end{cases}.$$

In this case the effort f(E,A) is defined by

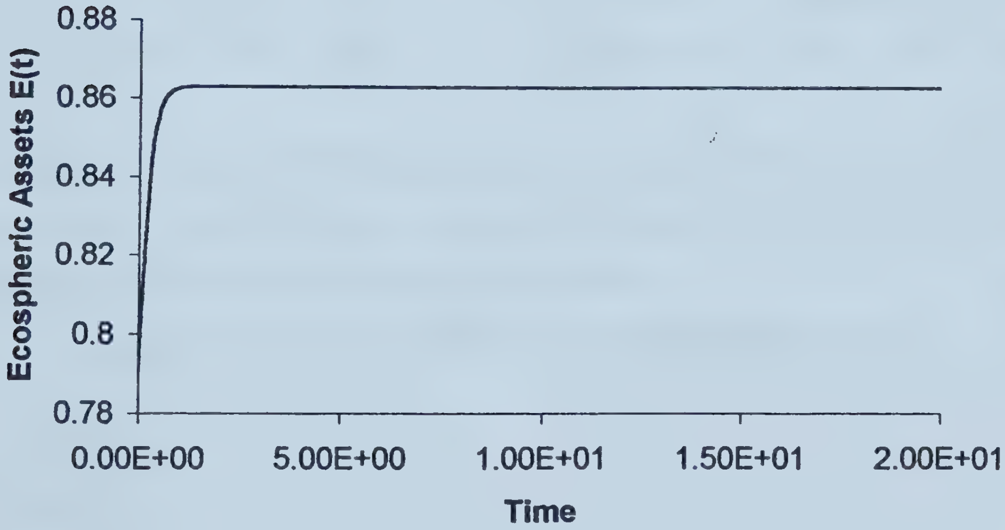
$$f(E,A) = f[E(\cdot)](t) = \begin{cases} \mu & \text{if } T_{3n} \leq t < T_{3n+1},\ n = 0,1,2,3,... \\ \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{min}} & \text{if } T_{3n+1} \leq t < T_{3n+2},\ n = 0,1,2,3,... \\ 0 & \text{if } T_{3n+2} \leq t < T_{3n+3},\ n = 0,1,2,3,..., \end{cases}.$$

Hence the modified ecospheric equation in this case is given by

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E + \mu & \text{if } T_{3n} \leq t < T_{3n+1},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{min}} & \text{if } T_{3n+1} \leq t < T_{3n+2},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E & \text{if } T_{3n+2} \leq t < T_{3n+3},\ n = 0,1,2,3,..., \end{cases} \quad (4.10)$$

A numerical solution to Equation (4.10) is as shown in Figure 4.12. We note that this solution is the same as the one shown in Figure 4.8 . Thus in this case the modified stop and the stop give the same result.

Figure 4.12: (Emax=0.89,Emin=0.84,Eo=0.79,mu=0.2)



4.4.2 $\mathrm{E}_{min} \leq E(0) < \mathrm{E}_{max}$

We define the k^{th} threshold by

$$\mathrm{k}^{\mathrm{th}} \quad \text{threshold} = \begin{cases} \mathrm{E}_{max} & \text{if } k = 3n+1,\ n = 0,1,2,3... \\ \mathrm{E}_{min} & \text{if } k = 3n+2,\ n = 0,1,2,3.... \\ \mathrm{E}_{min} & \text{if } k = 3n+3,\ n = 0,1,2,3,... \end{cases}$$

In the case the effort f(E,A) is defined by

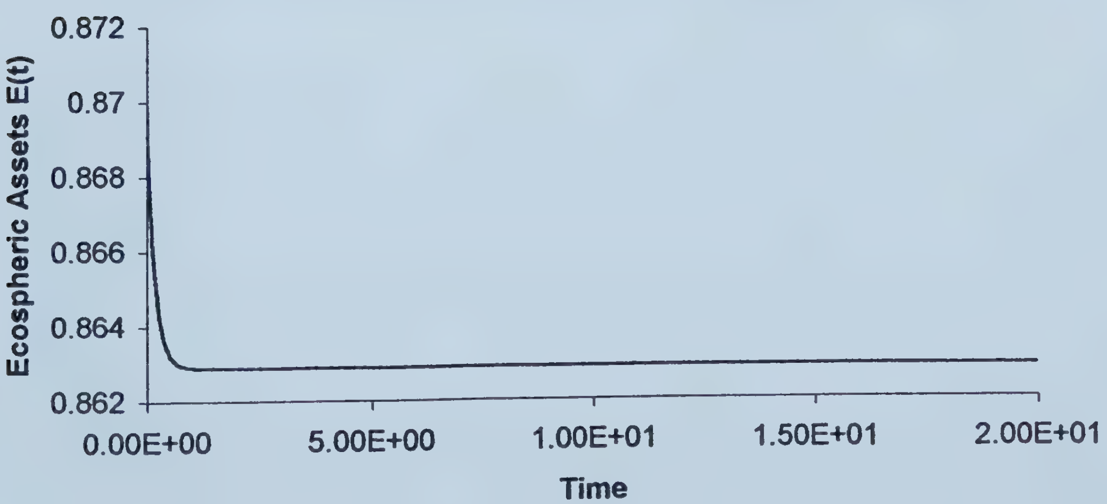
$$f(E,A) = f[E(\cdot)](t) = \begin{cases} \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{min}} & \text{if } T_{3n} \le t < T_{3n+1},\ n = 0,1,2,3,... \\ 0 & \text{if } T_{3n+1} \le t < T_{3n+2},\ n = 0,1,2,3,... \\ \mu & \text{if } T_{3n+2} \le t < T_{3n+3},\ n = 0,1,2,3,..., \end{cases}.$$

Hence the modified ecospheric equation in this case is given by

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E + \frac{\mu(\mathrm{E}_{max}-E)}{\mathrm{E}_{max}-\mathrm{E}_{min}} & \text{if } T_{3n} \le t < T_{3n+1},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E & \text{if } T_{3n+1} \le t < T_{3n+2},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \mu & \text{if } T_{3n+2} \le t < T_{3n+3},\ n = 0,1,2,3,..., \end{cases}. \tag{4.11}$$

A numerical solution to Equation (4.11) is as shown in Figure 4.13. We note that this solution is the same as the one shown in Figure 4.10. Thus in this case also the modified stop and the stop gives the same result.

Figure 4.13: (Emax=0.89, Emin=0.84,Eo=0.87,mu=0.2)



4.4.3 $E_{max} \leq E(0) \leq 1$

We define the kth threshold as

$$\text{k}^{\text{th}} \quad \text{threshold} = \begin{cases} E_{min} & \text{if } k = 3n+1,\ n = 0,1,2,3... \\ E_{min} & \text{if } k = 3n+2,\ n = 0,1,2,3.... \\ E_{max} & \text{if } k = 3n+3,\ n = 0,1,2,3,... \end{cases}.$$

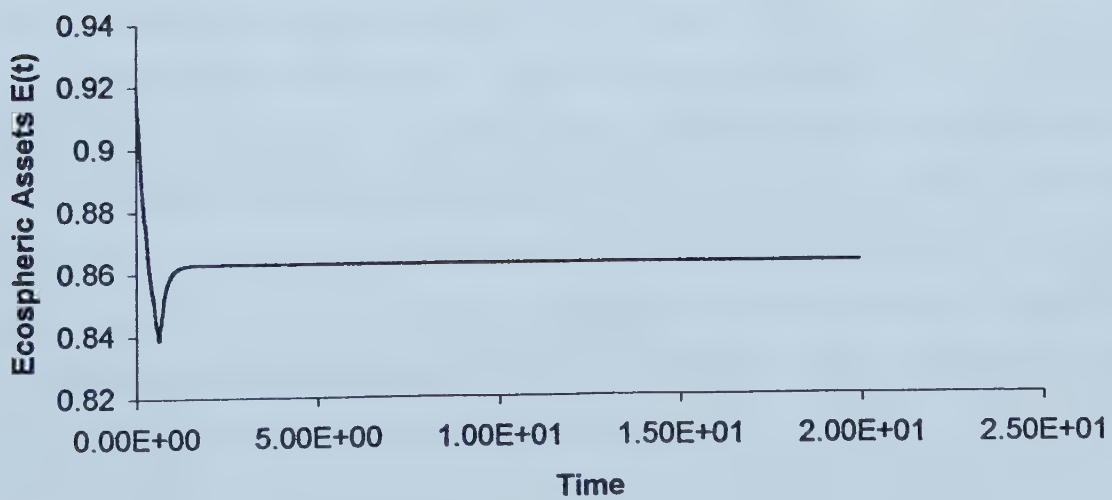
In this case the effort f(E,A) is defined by

$$f(E,A) = f[E(\cdot)](t) = \begin{cases} 0 & \text{if } T_{3n} \leq t < T_{3n+1},\ n = 0,1,2,3,... \\ \mu & \text{if } T_{3n+1} \leq t < T_{3n+2},\ n = 0,1,2,3,... \\ \frac{\mu(E_{max}-E)}{E_{max}-E_{min}} & \text{if } T_{3n+2} \leq t < T_{3n+3},\ n = 0,1,2,3,..., \end{cases}.$$

Hence the modified ecospheric equation in this case is given by

$$\frac{dE}{dt} = \begin{cases} -\kappa EA + \vartheta(1-E)E & \text{if } T_{3n} \leq t < T_{3n+1},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \mu & \text{if } T_{3n+1} \leq t < T_{3n+2},\ n = 0,1,2,3,... \\ -\kappa EA + \vartheta(1-E)E + \frac{\mu(E_{max}-E)}{E_{max}-E_{min}} & \text{if } T_{3n+2} \leq t < T_{3n+3},\ n = 0,1,2,3,..., \end{cases}. \tag{4.12}$$

Figure 4.14: (Emax=0.89, Emin=0.84, Eo=0.92,mu=0.2)



A numerical solution to Equation (4.12) is as shown in Figure 4.14. We note that this solution is different from the one obtained in Figure 4.11. In the case of the modified stop, we are able to raise the ecospheric assets to a value higher than E_{min} after it falls below E_{min}, whereas in the case of the stop, this rise does not occur.

4.5 Coupled System of Equations with a Simple Relay

In this section we solve the modified ecospheric equation with a simple relay coupled with the unmodified agricultural and industrial equations numerically. The results obtained in this case are compared with the results obtained in Chapter 3.

For comparism purposes, the values chosen in this section will be similar to those in the previous chapter. We choose

$$\alpha = 8.0, \quad \beta = 2.0, \quad \eta = 0.1, \quad \delta = 0.75, \quad \kappa = 0.5, \quad \vartheta = 2.0, \quad \mu = 0.2,$$

$$E_{min} = 0.82, \quad E_{max} = 0.89, \quad E(0) = 0, \quad I(0) = 0.8, \quad A(0) = 0.8.$$

The other parameter values are given for each figure below.

By comparing Figure 3.9 to 4.15, Figure 3.10 to 4.16, Figure 3.11 to 4.17, Figure 3.12 to 4.18, Figure 3.13 to 4.19, Figure 3.14 to 4.20, Figure 3.15 to 4.21, Figure 3.16 to 4.22, Figure 3.17 to 4.23, and Figure 3.18 to 4.24 , one could easily see that the solutions are similar, in fact qualitatively the same. Thus the introduction of a simple relay in the ecospheric equation doesn't change the general behaviour of the solution to the system without hysteresis. However, the hysteresis model minimizes the effort we put in to restore the ecosphere and hence the cost.

Figure 4.15:(Emax=0.89, Emin=0.82, gamma=0, xi=1)

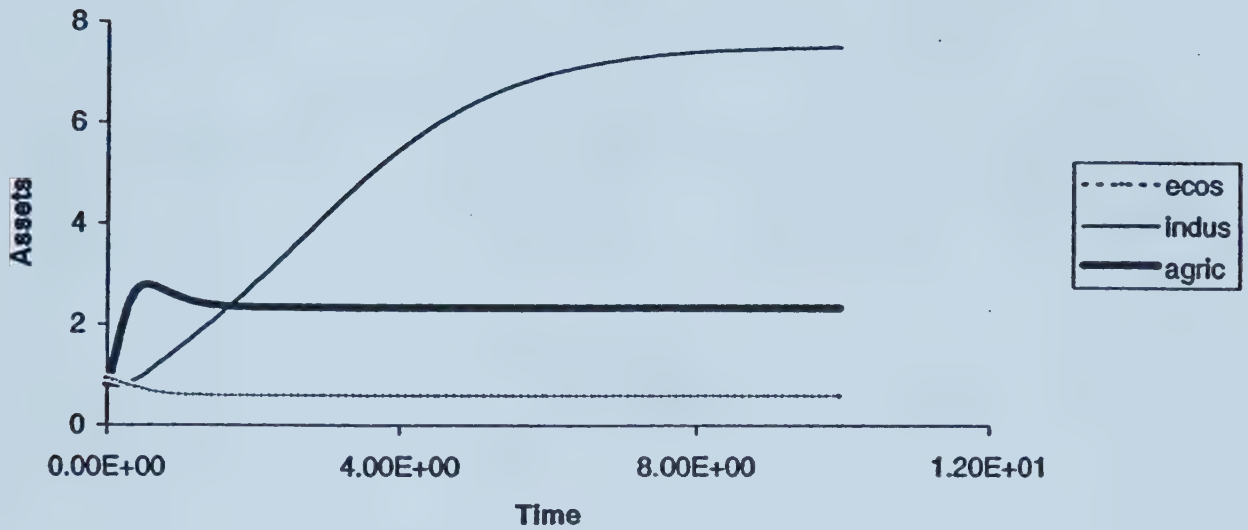


Figure 4.16: (Emax=0.89, Emin=0.82, gamma=0, xi=2)

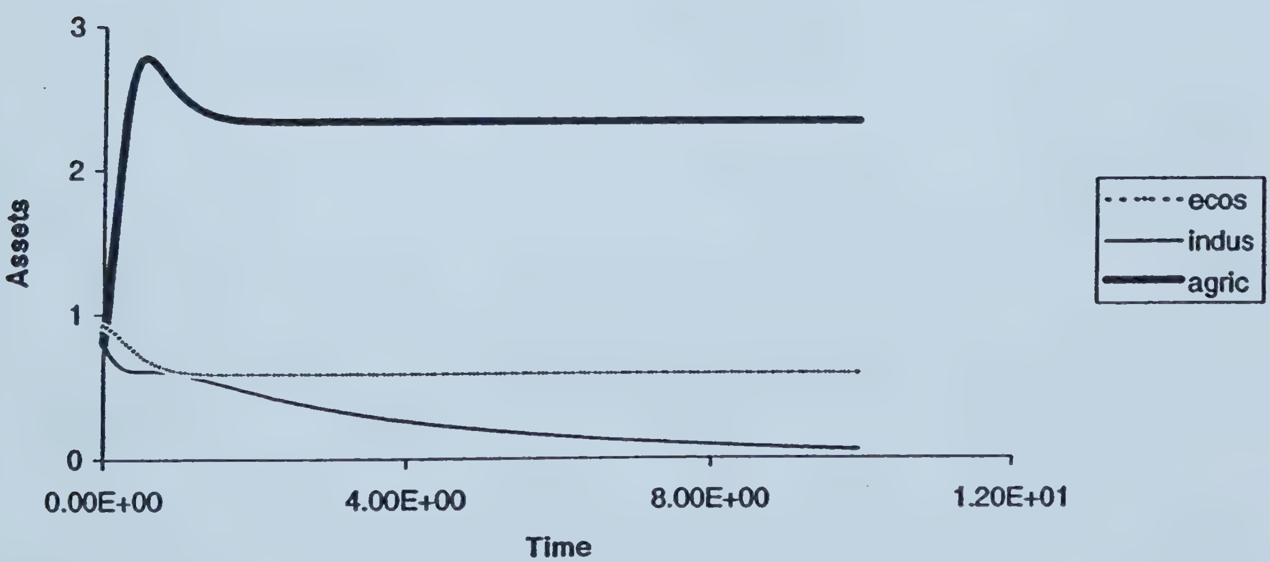


Figure 4.17: (Emax=0.89, Emin=0.82, gamma=-1, xi=1)

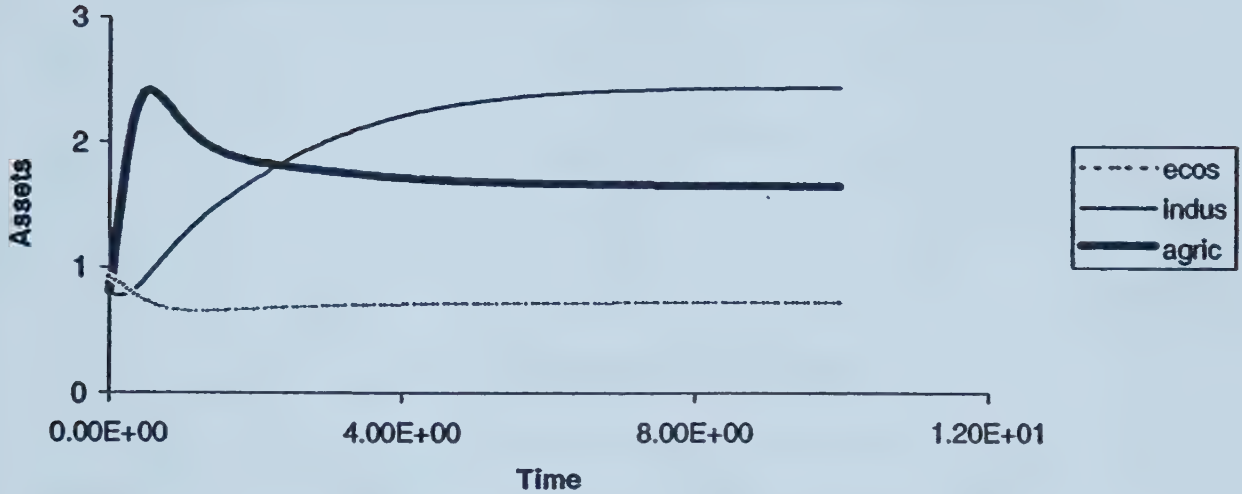


Figure 4.18: (Emax=0.89, Emin=0.82, gamma=-1, xi=2)

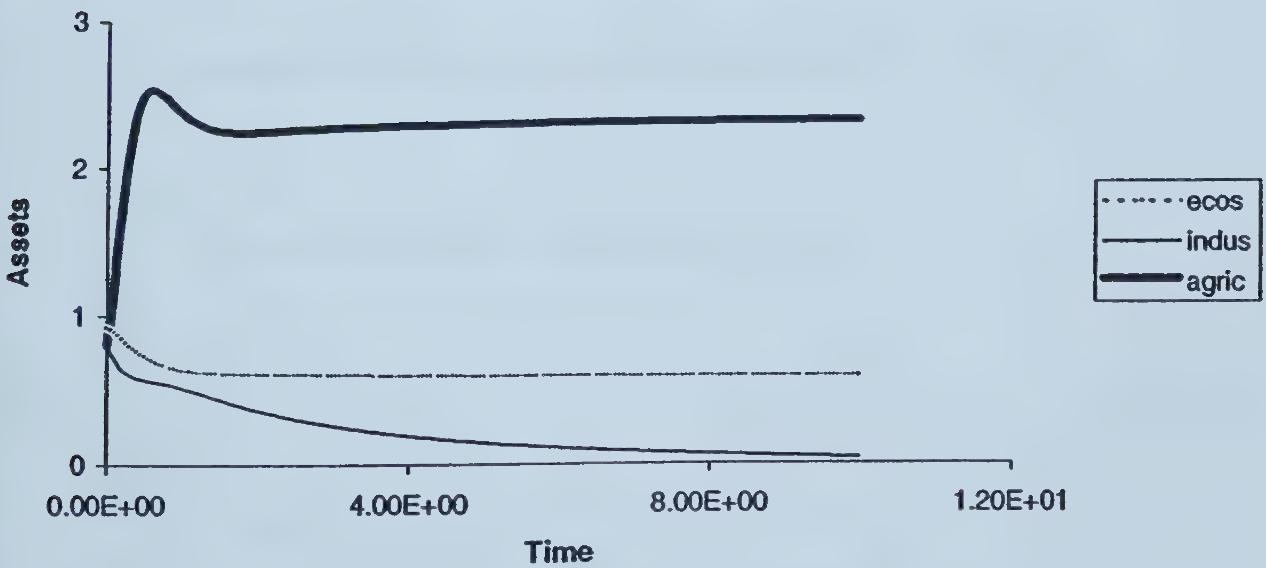


Figure 4.19: (Emax=0.89, Emin=0.82, gamma=0.2, xi=1.0)

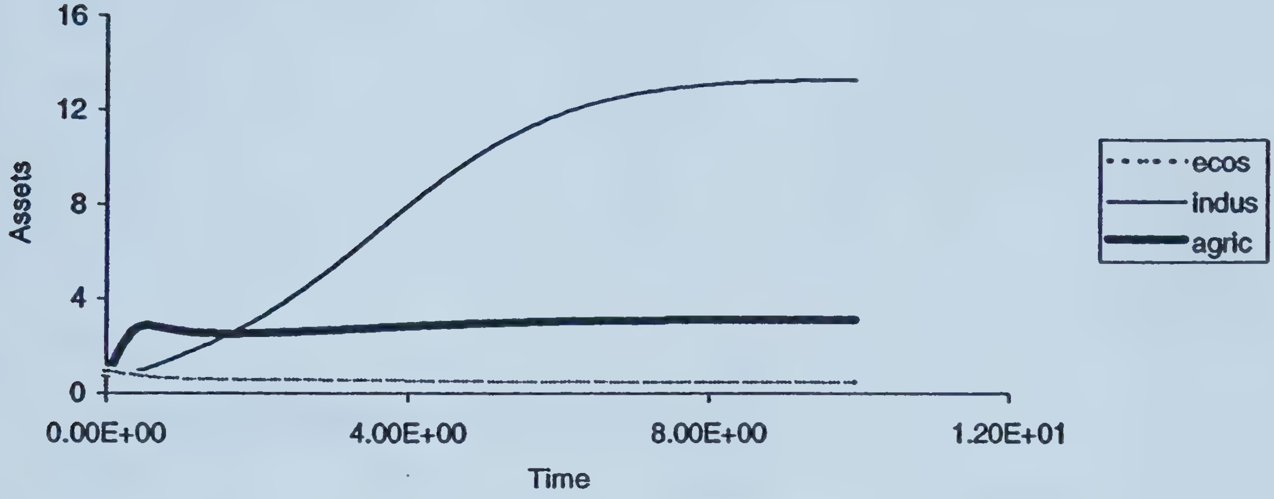


Figure 4.20: (Emax=0.89, Emin=0.82, gamma=0.2, Xi=2.0)

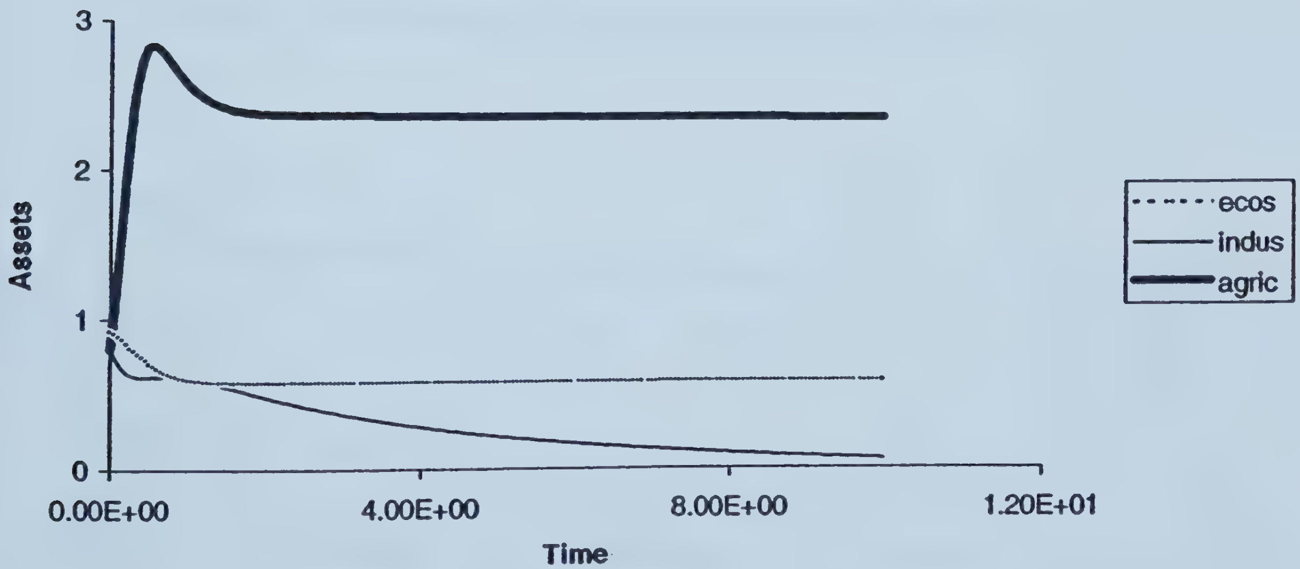


Figure 4.21: (Emax=0.89, Emin=0.82, gamma=1, xi=1)

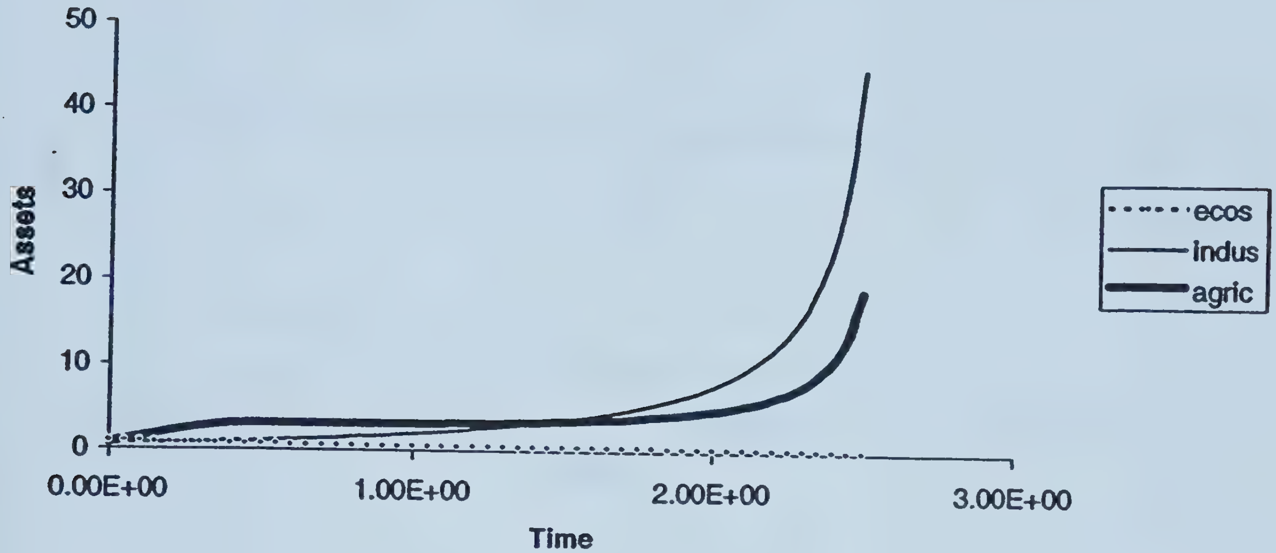


Figure 4.22:(Emax=0.89, Emin=0.82, gamma=1, xi=2)

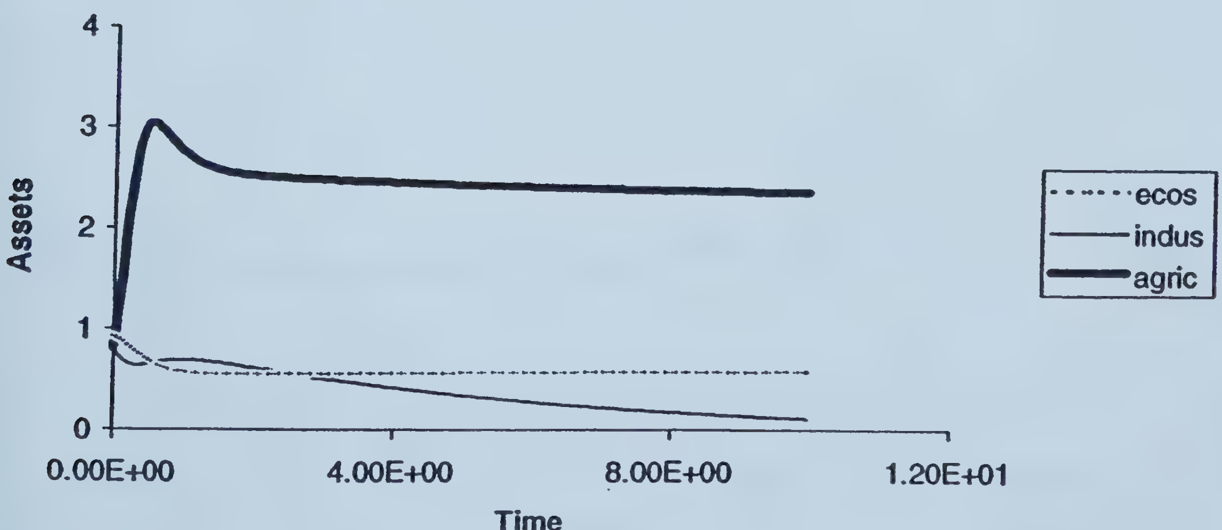


Figure 4.23: (Emax=0.89, Emin=0.82, gamma=0.5, xi=2)

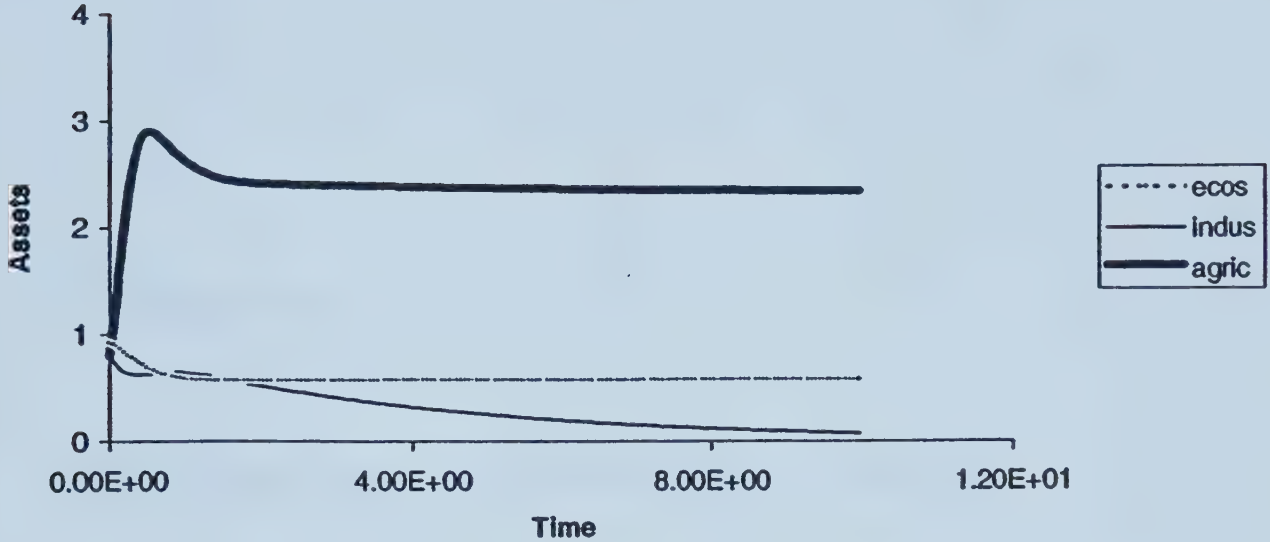
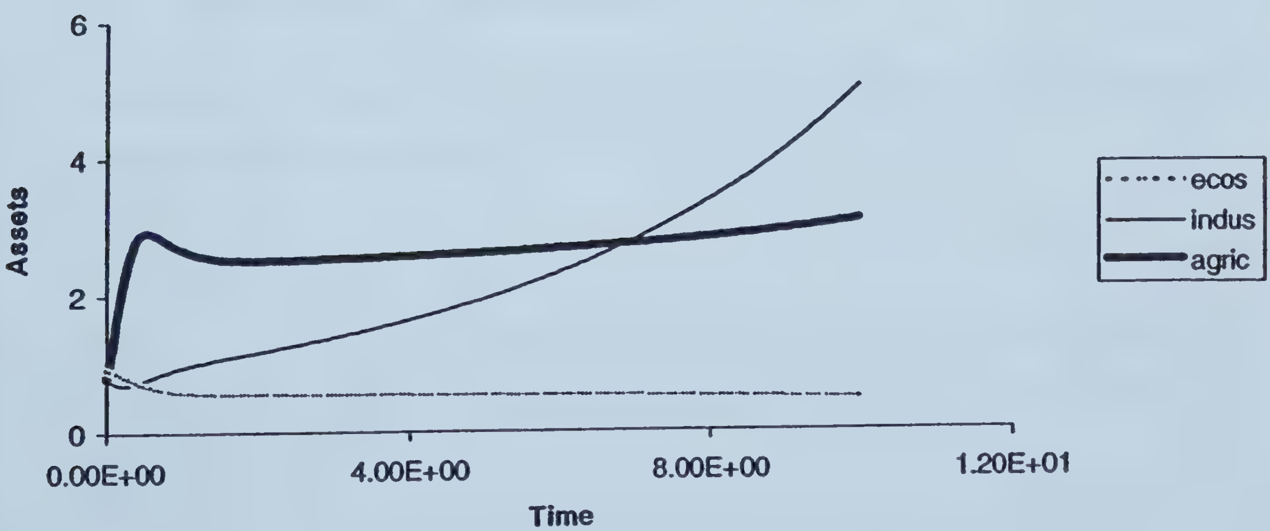


Figure 4.24: (Emax=0.89, Emin=0.82, gamma=0.5,xi=1.6)



Chapter 5

Discussion and Future Work

In this final chapter of the thesis we shall present a general summary of the results obtained in the previous chapters. Future analysis of the three dimensional system to include criteria for global stability will be outlined. Also future extension of the three dimensional model to include time delay in the ecospheric equation will be outlined.

5.1 Summary of Results

In this section, we shall present a summary of the most significant results obtained in the preceding three chapters.

5.1.1 Review of Chapter 2

In Chapter 2, we showed that if the ecosphere is in a state of equilibrium then the steady state $F_2 = (\alpha_0/\beta, 0)$ is globally asymptotically stable provided the following conditions are satisfied

$$(i)\quad \beta\eta \geq \gamma\delta, \quad (ii)\quad 4\beta\eta > (\gamma+\delta)^2, \quad \text{and} \quad (iii)\quad \alpha_0\delta < \beta\xi.$$

Thus industrial assets will go extinct while agriculture assets approach α_0/β. We also showed that the steady state $F_3 = (A^*, I^*)$ is globally asymptotically if

$$(i) \quad \beta\eta > \gamma\delta, \quad (ii) \quad 4\beta\eta > (\gamma+\delta)^2, \quad \text{and} \quad (iii') \quad \alpha_0\delta > \beta\xi.$$

are satisfied. Thus both industrial assets and ecospheric assets persist. These results show that the persistence or extinction of industrial assets depends on whether the product of the growth rates of industrial and agricultural assets is greater or less than the product of the diminishing returns coefficient for agriculture and the constant depreciation rate of industry. We observed that condition (i) is always satisfied if the interaction between industry and agriculture is that of parasitism or commensalism. We also emphasized that even if condition (ii) is not satisfied, it might be possible to have global stability of F_2 or F_3 under certain parametric configurations.

5.1.2 Review of Chapter 3

Using Equations (3.1), (3.2) and (3.3) and the associated assumptions we presented criteria for the local asymptotic stability of various non-axial planar and interior steady states. We showed that F_{2P} and F_{3P} (if they exist) are locally asymptotically stable with respect to solutions initiating from the interior of the (A,E)-plane, and that there are no nontrivial solutions to the system lying completely in the interior of the (A-E)-plane. F_{2P} exist if $\mu = 0$ and is generally locally asymptotically stable if $\frac{\alpha\delta\vartheta}{\kappa\alpha+\beta\vartheta} < \xi$. Also F_{3P} always exists if $\mu \neq 0$ and $\mu\beta \leq \kappa\alpha$ and is generally locally asymptotically stable if $\delta\alpha E^* < \beta\xi$. Thus the above conditions guarantee the local extinction of industrial assets.

It has been proved that if $\mu = 0$, $\delta\alpha\vartheta > \alpha\kappa\xi + \beta\vartheta\xi$ and $\gamma = 0 (\gamma < 0)$, then $F_{1I}(F_{2I})$ exists and is locally asymptotically stable. This shows that there will be

shares of assets among industry, ecosphere and agriculture and they will all survive the interaction locally. We also note that the local asymptotic stability of F_{1I} or F_{2I} implies that F_{2P} is unstable and the local asymptotic stability of F_{2P} implies the non-existence of F_{1I} or F_{2I}.

We also showed that for $\gamma > 0$ and $\mu > 0$, the state F_{2I} is locally asymptotically stable if

$$\frac{\alpha\vartheta\delta}{\xi} < \alpha\kappa + \beta\vartheta < \frac{\vartheta\gamma\delta}{\eta} \tag{5.1}$$

is satisfied. We could not however establish the local stability of F_{2I} if Equation (5.1) is reversed (i.e. $\frac{\alpha\vartheta\delta}{\xi} > \alpha\kappa + \beta\vartheta > \frac{\vartheta\gamma\delta}{\eta}$). However, under these conditions F_{2P} exists and is locally asymptotically stable, hence we conjecture that F_{2I} will be locally unstable.

It was established in Chapter 3 that if $\gamma = 0$, $\mu > 0$, and $\frac{\beta\xi}{\alpha\delta} < E^*_{1I} \leq 1$ then F_{3I} is locally asymptotically stable. It should be noted that $E^*_{1I} = E^*$ and hence the existence of F_{3I} implies F_{3P} is locally unstable and then the local asymptotic stability of F_{3P} implies the non-existence of F_{3I}. Thus if F_{3I} exists then there will be shares of assets among industry, agriculture and the ecosphere and they will all survive the interaction locally. We also showed that if $\mu > 0$ and $\beta\eta - \gamma\delta$ then F^+_{4I} is locally asymptotically stable. Similar conclusions for F_{3I} can be drawn for F^+_{4I} in this case. We also showed that F^-_{4I} (if it exists) is always unstable.

5.1.3 Review of Chapter 4

In Chapter 4, we introduced a hysteresis term into the ecosphere equation of the recovery model studied in Chapter 3. Four different types of the hysteresis terms were considered. These were the switch case, simple relay case, stop case and the modified stop case. In each of the cases, the modified ecospheric equation was solved by

computer simulation for various parameter configurations, assuming that agricultural assets remained constant. Various graphs depicting these solutions were presented. We later coupled this modified ecospheric equation with the unmodified agricultural and industrial equations and solve this three dimensional system numerically in the case of the relay. The graphs for the various solutions for various parameter configurations were presented. These solutions were compared with the solutions from Chapter 3, and it was observed that the solutions are qualitatively the same.

5.2 Conclusions

The model in this thesis is designed to investigate the future of agriculture and industry as we continue to pressure the ecosphere. From our model, we observed that if we assume the ecosphere is in the state of equilibrium then we can always choose our parameters such that both industry and agriculture will never go extinct. On the other hand we can also choose the parameters such that industry will always go extinct and only agriculture will survive the interaction. Since our aim in this thesis is generally based on asset creation rather than profit making or maximization of assets we will prefer the former case to the later.

The results obtained in Chapters 3 and 4 also shows that depending on how we choose our parameters, all the three assets may survive the interaction or only two may survive. Agriculture will always survive the interaction, but the survival of industry and/or the ecosphere depends critically on how we choose the parameters. The most interesting results to me are in Figures 3.6, 3.15, 3.18 and 4.17, where industrial and agricultural assets grow unbounded even as the ecospheric assets become extinct. Thus if we choose our parameters such that such solutions are obtained, then basically,

these results tells us that we need not worry about the the state of the ecosphere if we are only interested in maximizing industrial and agricultural assets. We must however, state that this may not occur in real practice.

Lastly, we state that since results obtained in the hysteresis model and the non-hysteresis model are qualitatively the same, one would prefer using the hysteresis approach in real life practice instead of the non-hysteresis case, because this form deals with minimizing the effort, and hence cost, in restoring the ecosphere.

5.3 Future Work

In our future work, we will try to find criteria for the global asymptotic stability of the locally asymptotic non-axial planar and interior steady states of the ecospheric recovery model. Since in real life, the addition of environmental quality such as nutrient to the ecosphere to compensate for the loss in ecosphere assets as a result of it use for agricultural activities will take a certain amount of time, say τ to convert into usable ecospheric assets, our model will therefore be more realistic if we in co-operate a time delay into the ecospheric equation. Thus the addition of time delay into the ecospheric equation will be the next step in our model.

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Appendix A: Flow Charts

Chart 1: Flow Chart for Ecospheric Equation with a Simple Relay

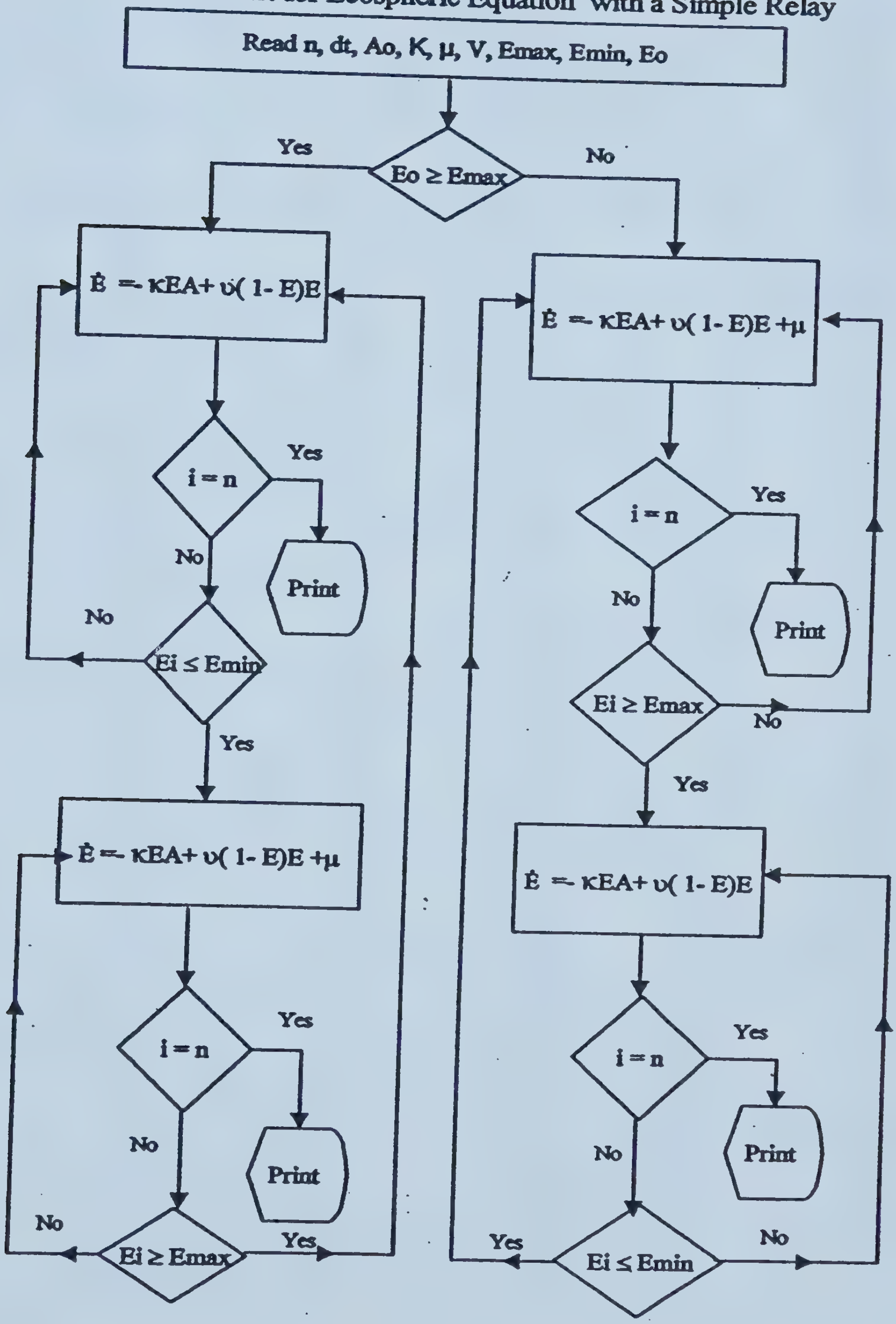

Chart 2. Flow Chart for Ecospheric Equation with a Stop

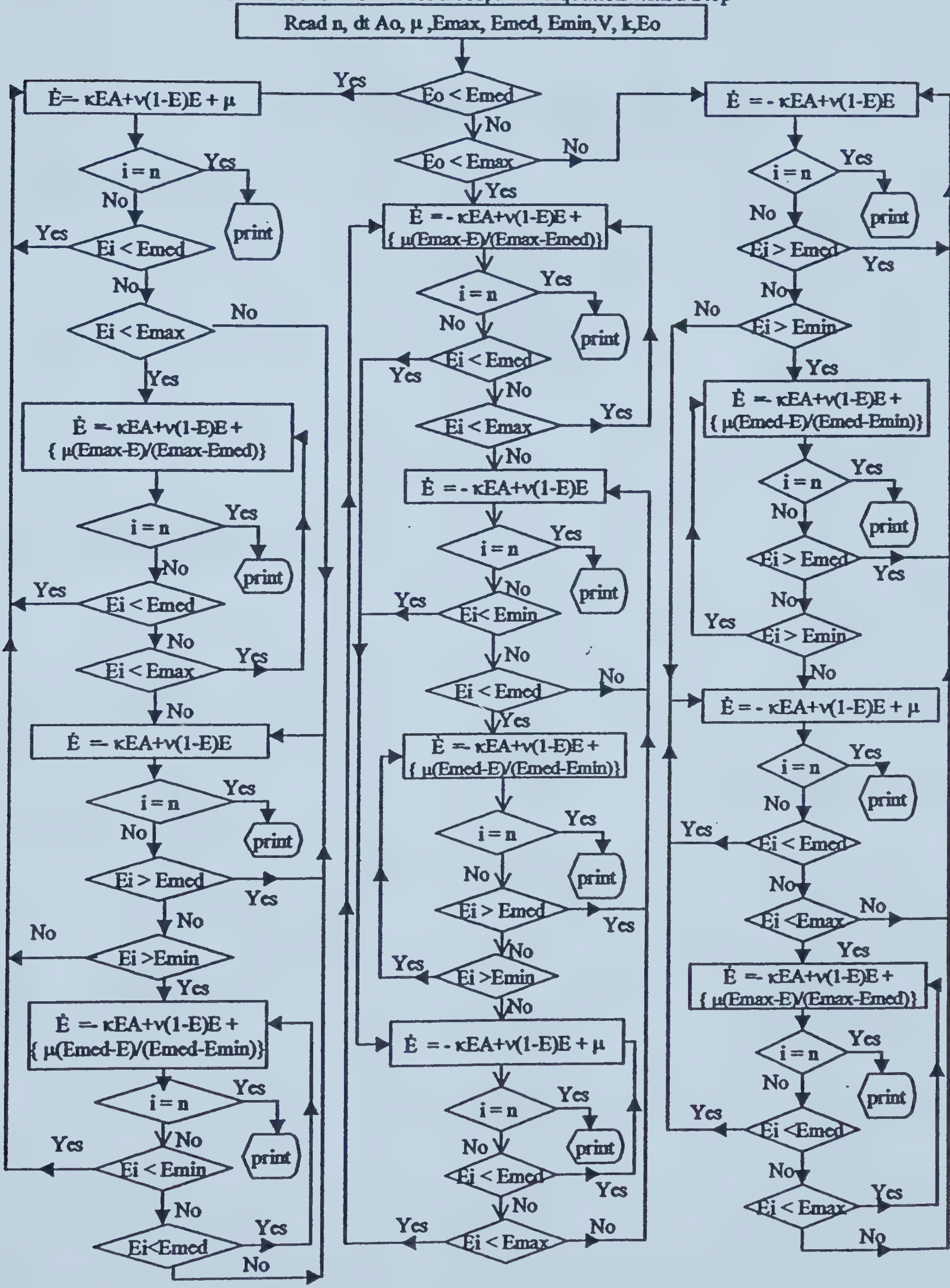

Chart 3: Flow Chart for Ecospheric Equation with modified Stop.